

How to Integrate Almost Anything

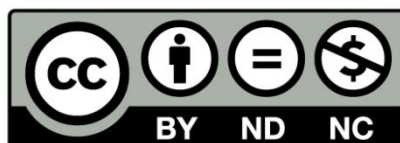
A Detailed Explanation of the Techniques of Integration

by
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$$\int u \frac{dv}{dx} dx$$

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This is version 1.0 of the book.

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The author regretfully admits to being only human, and would be grateful to hear of any errors so that they can be corrected.

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Introduction

1 Why This Book?

I have always been very fond of integration. While receiving some rather sketchy teaching of the process at school, where it was explained as merely an inverse process to differentiation, I spent a couple of evenings carefully reading my textbook's very full account of the "meaning" of integration, and found it revelatory.

Subsequently, I was dragged in a most perfunctory fashion, under the pressure of an impending exam, through a hasty exposition of the various techniques which were available to integrate functions which would not respond to the simple rules I had so far learned. It was, indeed, only some time later that I was able to acquaint myself more thoroughly with this material

What I have set out to do in writing this book has been to produce the textbook I wish I had when I was trying to learn about the techniques of integration myself. Accordingly, I have filled the book with over a hundred worked examples illustrating each of the techniques I have chosen to address, and I have tried to offer detailed explanations at every turn. I have also presented a number of algorithmic sets of instructions for the execution of many of the techniques explained.

Clearly, the material I am presenting here is not new, and indeed there are plenty of resources, both printed and on-line, to help students learn the techniques of integration. However, in researching the book, I have found these to be spread out over a plethora of different locations, and so I have tried here to gather together in one resource as much information about the techniques of integration as I could assemble within a reasonable time.

I hope that a combination of comprehensive coverage, quantity of worked examples and the care with which I have tried to explain everything will make this a useful addition to the extensive literature on calculus which is already available, as well as a useful and functional tool which students will find highly relevant to their needs.

I have chosen not to include sets of practice examples. This is because the text is already quite lengthy, and also because practice examples are available in their multitudes from other sources, so that it would be rather pointless to add to their number or repeat existing examples here. There are, of course, as any student mathematician will agree, few things more fatuous than practice examples in a textbook which are rendered almost impossible for most readers by the conciseness or incompleteness of the explanation which has preceded them

It will be obvious to those who make their way through the entire book that there are some more advanced techniques which I have not covered in the current edition – the tangent half-angle substitution is a case in point. Additionally, I have chosen not to address the various interesting issues associated with the hyperbolic functions in the context of integration. It was clear to me, however, when writing the material that I would have to stop somewhere, if only to make a finished product of some sort available. A (hypothetical) future second edition might remedy some of the more glaring omissions.

2 How to Use This Book

I envisage at least two ways in which you might make use of this book.

First, if you are learning about the techniques of integration or if you want to reacquaint yourself with them, you may like to start at the beginning of [Part 1](#) and make your way right through to the end of [Part 3](#), omitting any sections you find less useful, and consulting the appendices when a reference to them piques your interest.

Second, you can use it simply as a reference book, by consulting the full index of integrals at the end to find something which looks a bit like whatever knotty integration problem you may be facing. You can then go straight to the relevant part of the text, and follow any back-references you may find there if necessary for greater understanding.

The book will, I hope, be of particular use to anyone studying mathematics in their final years at school or their earlier years at university. Students of any of the mathematical sciences, such as physics or statistics, as well as engineering, may also find it useful.

It may be worth pointing out in addition that the book contains a fairly full explanation of the algebraic Method of Partial Fractions (which doesn't seem to be as widely or as thoroughly taught as it once was) as well as my own tabular approach to polynomial division, which I hope will prove a labour-saving approach for those who are used to having to wade through the horrid "long division" technique.

3 What You Need to Know to Use This Book

This book is concerned almost exclusively with integration – as such, I haven't spent any time on explaining how to differentiate. You need to know how to differentiate, including how to use the Chain Rule, the Product Rule and the Quotient Rule in order to get the most out of the book.

Additionally, I have concentrated on particular techniques for integration, including substitution, integration by parts and the application of various algebraic and trigonometric methods. I have assumed that you will already know and be able to apply the following basic rules of integration:

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

You should also know how to integrate term by term where appropriate – in other words, you should be able to apply the rule:

$$\int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx$$

and you should also be aware of how we can take a constant coefficient outside the integral sign, as illustrated by the rule:

$$\int kf(x) \, dx = k \int f(x) \, dx$$

The book does contain quite a lot of algebraic manipulation, so it will certainly pay not to be afraid of this essential mathematical skill. I have tried to explain any unusual algebraic “tricks” I have had to call on where these prove necessary. In particular, you will find [Part 3](#) much more enjoyable if you have at least some familiarity with solving systems of linear equations (even just two equations in two variables).

Two final points are worth making. First, some familiarity with the behaviour of the basic trigonometric functions $\sin x$ and $\cos x$ will be very useful: I have tried to give quick explanations of many of the trigonometric identities the book calls upon. Second, I have made occasional use of the Binomial Theorem – though in all cases I have explicitly quoted the particular instance needed whenever the theorem has been used.

4 The Content of the Book

When we learn to differentiate, we quickly become familiar with a number of techniques, including the Chain Rule and the Product and Quotient Rules, which allow us to write down the derivatives of a wide variety of functions without much difficulty. Admittedly, these techniques of differentiation usually produce as results some fairly complicated functions by way of output.

For example, the Chain Rule (in its simplest form) takes a function formed by the composition of one function on another and gives as its result the product of two functions. In other words, we can say that, using the Chain Rule, differentiation takes a functional composition into a product of functions.

The Product Rule takes the product of two functions into the sum of two functional products. The Quotient Rule starts with the quotient of two functions and produces a new quotient, with a numerator comprising the difference of two functional products and a denominator comprising the square of the function in the original denominator.

When it comes to learning integration, however, we don’t get a similar set of tools, and we rapidly encounter functions which cannot be integrated simply. Indeed, it seems that there are even some functions whose indefinite integrals cannot be expressed using the various combinations of “elementary” functions which are available to us.

There are, however, some venerable and effective techniques which can be applied to the problem of integrating “difficult” functions, and I hope to explain and examine these techniques in some detail in this book.

To start with, I will look at the problem of integrating one example of a “difficult” function, and use it as our starting point for a close look at the technique known as *integration by substitution*.

I will then proceed to examine a second technique, called *integration by parts*.

For both of these methods, I will discuss several examples of their use, and then try to set out some useful recommendations on when each method is the best approach, and how exactly we should decide the best way to apply it.

In the final part of the book, I present a number of worked examples of the use of algebraic and trigonometric techniques to put integrands into a form where we can use either the standard rules of integration or one or some of the techniques already discussed to integrate them. I also show some examples of the use of a combination of several different techniques to evaluate integrals.

5 A Note on the Constant of Integration

From the first stages of learning to integrate, we are taught that when writing out indefinite integrals, we should always add a constant term, as follows:

$$\int x^2 dx = \frac{1}{3}x^3 + C$$

The simple justification usually offered for this practice is that there are infinitely many possible “versions” of the indefinite integral, each of which has a different constant term. In the case I have chosen, therefore, it would be fair to say that three randomly chosen candidates as the indefinite integral of $f(x) = x^2$ are:

$$\frac{1}{3}x^3 + 4, \frac{1}{3}x^3 + \frac{5}{16} \text{ and } \frac{1}{3}x^3 - 15$$

Treating the indefinite integral $F(x)$ of our chosen function $f(x)$ as the *antiderivative* of that function – so that $F'(x) = f(x)$ – we can easily see that any of the candidates I have listed above (and any others you may care to choose) all differentiate to give $f(x)$. This is because the derivative of a constant term is always equal to zero.

We therefore add our constant term to an indefinite integral to show that we are aware that adding any constant term to the expression in x which we have obtained will provide an equally valid solution. In applications of integration to practical situations, we may well have at our disposal some additional information (for example, the value of the integral when $x = 0$) which will allow us to assign a particular value to the “constant of integration” which I have labelled C .

In order to make clear that the constant term we routinely add to our indefinite integrals can represent any actual constant term which may arise from the circumstances in which we need

Introduction

to use the result of integration, some writers choose not to use a particular variable, but instead use an abbreviated narrative description, such as:

$$\int x^2 dx = \frac{1}{3}x^3 + \text{const}$$

Others choose a different letter as the place-holder for the constant term, such as:

$$\int x^2 dx = \frac{1}{3}x^3 + k$$

I have chosen in this book to use a capital C throughout – partly, I admit, because this is the notation I first learned as a teenager in the Scottish education system, and partly because no notation I have encountered since seems to offer advantages convincing enough to make me want to change. In particular, I have sometimes used the letter k for other purposes in this book, so I have been at pains to avoid creating confusion by making it stand for the constant of integration as well.

I have been a little naive in using the same letter C as the constant term resulting from the integration of an expression under substitution and the constant term in the indefinite integral after reversing the substitution. I have avoided burdening the reader with a theoretical exploration of the rightness or wrongness of this convention, because this book is meant to explain techniques of integration, and not the theoretical underpinnings of the process of integration itself.

The only occasions on which I have used a different symbol for a constant of integration are those where the mechanical process of integration gives rise to a constant term in the result it produces. Here, I have generally discussed the usefulness of “restating the constant”, and have used the symbol C' for the “restated” constant, in order to emphasise that it subsumes an explicit constant which has arisen from the integration process.

Part 1

Integration by Substitution

1 A “Difficult” Function

I’m going to start by considering the following indefinite integral, which does not seem amenable to the simple rules for integration we are taught when beginning calculus:

$$I = \int \sin x \cos x \, dx$$

I first encountered this integral along with some other similar ones, when reading about equations which represent sounds in terms of their harmonics. We might perhaps expect to be able to handle the integration of a simple product of two trigonometric functions without breaking too much of a sweat. We can certainly differentiate the product easily enough, using the Product Rule. But how exactly should we go about integrating it?

One thing might occur to us at the outset: this is a product of two elementary functions, both of which we know how to integrate on their own. We don’t, however, have a Product Rule for integration at our disposal, so we might next recall the Chain Rule, which allows us to differentiate functions which are the result of the operation known as *functional composition*. This rule takes a “function of a function” and produces as its derivative the *product* of two or more functions. Using the “dashed” notation, we recall that if

$$F(x) = g(f(x))$$

then its derivative will be:

$$F'(x) = g'(f(x)) \cdot f'(x)$$

which is a product of two functions.

Now, the Fundamental Theorem of Calculus tells us that the integral of $F'(x)$ is quite simply $F(x)$ - indeed, this function is sometimes referred to as the “antiderivative” of $F'(x)$ (we have, of course, left out the famous “constant of integration”). So if we can shoehorn the expression $\sin x \cos x$ into the shape given in general terms above for a derivative according to the Chain Rule, we might be home and dry.

To start with, then, let’s set $f'(x) = \cos x$. Our knowledge of differentiation should allow us to deduce straight away that $f(x) = \sin x$. Continuing to fit our “difficult” function into a Chain Rule template, we then have $g'(\sin x) = \sin x$, from which we can state more generally that $g'(x) = x$. To find the function g , then, we just have to integrate the right-hand side of this expression for $g'(x)$, and this brings us to the conclusion that:

$$g(x) = \frac{1}{2}x^2 + C$$

Putting together our conclusions about the functions f and g and using functional composition - recalling that $F(x) = g(f(x))$ -, we can finally say that:

$$I = g(f(x)) = \frac{1}{2}\sin^2 x + C$$

And so, by managing to fit our “difficult” function into the structure of a derivative according to the Chain Rule, we find that we can write down its integral after all.

2 Integration by Substitution

Now it’s time to explain the first of the techniques for integration of “difficult” functions which form the subject of this book. Integration by substitution – the name of this first technique - is simply a slightly more mechanical way of following through the logic based on the Chain Rule for differentiation which we used in the previous section.

I’ll start by taking the function we integrated in [Section 1](#) as our first example. But before doing so, a word or two will be necessary about the principles behind this technique. The starting point is as follows: if we are asked to integrate a function $f(x)$ with respect to x , and the function appears too complicated for our simple rules of integration to handle, then we could try to turn the function into a simpler one which we can handle by making a *substitution*. This act of substitution would involve “packing up” some of the complexity in a helper function $u = h(x)$ and getting into a position where we can integrate with respect to u . You will find that the usual choice of alternative variable in this context is the letter u - in fact, the substitution technique we are examining here is often referred to as “ u -substitution”.

How do we get into this desired position? Well, let’s first remark that if we are trying to evaluate the indefinite integral

$$I = \int f(x)dx$$

then making our substitution in the way I have described above will transform this into the new integral

$$\int g(u)du$$

We would, of course, hope that the new function $g(u)$ is rather simpler than $f(x)$ - perhaps a polynomial in u , which we can integrate more easily than the function we began with. So far, so good – however, the presence of the symbol dx in the integral means that we have not yet got into a position where we are integrating with respect to u - so there is still work to be done.

We can get to the desired position by performing something of a “conjuring trick” – we can, in fact, claim the following identity to be true:

Integration by Substitution

$$du = \frac{du}{dx} dx$$

where the instances of dx are “cancelled”, just as if we were dealing with an ordinary fraction.

This piece of notational legerdemain brings our indefinite integral into the form:

$$\int g(u) \frac{du}{dx} dx = \int g(u) du$$

And so we find ourselves able to integrate with respect to u , which is the position we wanted to reach.

In order to put this theoretical apparatus into practice, let's return to our initial problem, to evaluate the indefinite integral

$$I = \int \sin x \cos x dx$$

and walk through an application of our new technique to find its solution.

If we start by making the substitution $u = \sin x$, and immediately note that as a consequence:

$$\frac{du}{dx} = \cos x$$

we find that by the reasoning we have just set out, our indefinite integral is transformed into the simpler integral

$$\int u \frac{du}{dx} dx = \int u du$$

We know that this integrates simply and easily to:

$$\frac{1}{2} u^2 + C = \frac{1}{2} \sin^2 x + C$$

undoing the substitution once the hard work has been done, in order to provide us with a result expressed in terms of x , just like our original function.

And thus we arrive at exactly the same result that our informal Chain Rule reasoning in [Section 1](#) gave us.

3 Some Further Examples

A good way to get an appreciation of how to use this technique of integration is to examine a number of examples of its use, and I will now set out several which will illustrate the method

Integration by Substitution

in action. These will also allow us to proceed to make some generalizations and create a set of rules for using the method.

Example 1

Let's consider the following indefinite integral:

$$I = \int \cos x (1 + \sin x + \sin^2 x) dx$$

Although this looks daunting, it's only a slightly more complicated version of our original example, and it can be solved in a similar way. As before, we will make the substitution $u = \sin x$, and immediately note the useful consequence:

$$\frac{du}{dx} = \cos x$$

Our indefinite integral is now transformed into:

$$\int (1 + u + u^2) \frac{du}{dx} dx = \int (1 + u + u^2) du$$

This integrates very simply to:

$$u + \frac{1}{2}u^2 + \frac{1}{3}u^3 + C = \sin x + \frac{1}{2}\sin^2 x + \frac{1}{3}\sin^3 x + C$$

reversing the substitution to obtain a result in terms of x . As a final step, we can if desired factorise out $\sin x$, to obtain the expression:

$$\sin x \left(1 + \frac{1}{2}\sin x + \frac{1}{3}\sin^2 x \right) + C$$

This brings the solution into the same shape as that in which our integrand was presented, although it's probably fair to say that the decision on whether or not to factorise is largely a matter of taste and convenience in the particular context in which the problem arises.

Example 2

Here we will look at the following indefinite integral:

$$I = \int (\sin x + \cos x)(\sin x - \cos x)^2 dx$$

With this integral, if we start by asking ourselves what kind of simplification would make our job easier, we might remark that since we already know how to integrate u^2 , we could try a substitution of $u = \sin x - \cos x$. This substitution gives us the derivative:

$$\frac{du}{dx} = \cos x + \sin x$$

which is, of course, equal to the expression $\sin x + \cos x$, which is the first factor in the integral we need to evaluate. We can therefore transform our integral into:

$$\int u^2 \frac{du}{dx} dx = \int u^2 du$$

This integrates to:

$$\frac{1}{3}u^3 + C = \frac{1}{3}(\sin x - \cos x)^3 + C$$

on reversing the substitution.

Note

All the examples we have looked at so far have used simple trigonometric functions. These have appealed to me in introducing integration by substitution because of their agreeable symmetry under differentiation, which makes it easy to see how a particular substitution might help us. It is now time, however, to examine the technique at work with non-trigonometric functions, and at the same time to introduce a more algorithmic approach to its use.

Example 3

Consider the following indefinite integral:

$$I = \int x(x^2 + 1)^2 dx$$

Of course, this is actually simple enough to let us multiply out the squared term and then multiply the result by x , ending up with a simple polynomial in x which we could integrate term by term. However, we will turn our backs on this tempting approach and try to apply our substitution technique to the problem of integrating the function “as is”. Who knows, we might even save some labour by doing this.

Well, to start with, we know how to integrate u^2 , so we might try a substitution of $u = x^2 + 1$. This gives us a derivative:

$$\frac{du}{dx} = 2x$$

Unfortunately, unlike the preceding “convenient” examples, we don’t have a $2x$ term which we can straightforwardly replace in the way we have been doing so far.

In order to get round this difficulty, we are asked to perform a little more Leibnizian hand-waving and manipulate du/dx as if it were a fraction. This manipulation gives us:

$$du = 2x dx$$

Integration by Substitution

or alternatively:

$$\frac{1}{2} du = x dx$$

We do have an $x dx$ term in the written expression of our integral, so we can substitute this with $(1/2)du$. This means that, with our chosen substitution, the integral can be written as:

$$\int \frac{1}{2} u^2 du$$

which integrates to :

$$\frac{1}{6} u^3 + C = \frac{1}{6} (x^2 + 1)^3 + C$$

on reversing the substitution.

Note

In the previous example, when we were manipulating du/dx , we could have gone the whole hog and obtained an expression for dx - this would have been:

$$dx = \frac{1}{2x} du$$

On substituting this back into our integral, as well as making the original substitution we decided to try, the integral would be transformed into:

$$\int x \cdot u^2 \cdot \frac{1}{2x} du$$

We can now see that the x in the denominator of fractional factor would cancel out with the x at the start of the integrand, thus giving us the same neat integral in terms of u which we obtained by a slightly different route. We will use this technique of obtaining an expression for dx in the next few examples, and this will eventually allow us to set out a fairly neat algorithm for integration by substitution.

Example 4

Now consider this indefinite integral:

$$I = \int x\sqrt{1-3x^2}dx$$

In line with our reasoning in the examples above, we immediately note that we know how to integrate $\sqrt{u}$, or $u^{1/2}$, so that would suggest a substitution of $u = 1 - 3x^2$. Let's now see what this gives us by way of an expression for the symbol dx . We can write:

Integration by Substitution

$$\frac{du}{dx} = -6x$$

so that:

$$dx = -\frac{1}{6x} du$$

Plugging these expressions back into our original integral (and cancelling the x at the start of the integrand with the x in the denominator of the expression for dx) gives us :

$$\int x \cdot u^{1/2} \cdot \left(-\frac{1}{6x}\right) du \text{ which simplifies to } \int -\frac{1}{6} u^{1/2} du$$

and this integrates to:

$$-\frac{1}{9} u^{3/2} + C = -\frac{1}{9} (1 - 3x^2)^{3/2} + C$$

on reversing the substitution. Swapping back from our “rational power” notation to restore the square root sign we originally had in our integrand would give us this solution:

$$I = -\frac{1}{9} \sqrt{(1 - 3x^2)^3} + C$$

Example 5

Here, we are going to look at another indefinite integral which may appear slightly less complicated than some of our previous examples.

$$I = \int \cos(2x + 3) dx$$

We might first note, however, that, unlike our previous examples, this is not expressed as a product of two functions, but is instead a rather simpler “function of a function” – and this is why it may seem less complicated in appearance. It is not readily amenable to integration using the basic rules, though, so we will have to try our substitution technique.

Following the intuitive reasoning used in earlier examples, we notice that we do know how to integrate $\cos u$, which points us towards a substitution of $u = 2x + 3$. We can then write:

$$\frac{du}{dx} = 2$$

so that:

$$dx = \frac{1}{2} du$$

Putting these expressions back into the original integral, we obtain:

Integration by Substitution

$$\int \frac{1}{2} \cos u \, du$$

which integrates to:

$$\frac{1}{2} \sin u + C = \frac{1}{2} \sin(2x + 3) + C$$

on reversing the substitution.

Note

The last example is one of a class of indefinite integrals which will respond to the use of a substitution of the form $u = ax + b$. These are particularly interesting since they allow us to use substitution to handle the integration of functions of the form $f(ax + b)$. Using the substitution proposed, we will always obtain a very simple expression for dx , namely:

$$dx = \frac{1}{a} du$$

This will simplify our job to one of integrating

$$\frac{1}{a} f(u)$$

with respect to u . If f is an easily-handled function, then the introduction of a constant coefficient will not make our job any harder.

Example 6

We will conclude this set of illustrative examples with one which looks a little different, but not too much so for comfort:

$$I = \int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$$

Since we can readily integrate $\sin u$, we might consider trying the substitution $u = \sqrt{x}$, otherwise written as $u = x^{1/2}$. This gives us:

$$\frac{du}{dx} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$$

so that:

$$dx = 2\sqrt{x} du$$

When we plug these expressions back into the original integral, the $\sqrt{x}$ in the expression for dx and the $\sqrt{x}$ in the denominator of the original function cancel each other, so that we now

only have to integrate:

$$\int 2 \sin u \, du$$

which integrates to:

$$-2 \cos u + C = -2 \cos \sqrt{x} + C$$

on reversing the substitution.

4 An Algorithm for Integration by Substitution

Through force of repetition in the above set of examples, it should now be possible to see our way to writing an algorithm which can be used every time integration by substitution, in the basic form we have seen so far, can be applied. By abstraction from the examples, which became more and more “algorithmic” as we worked through them, our procedure should run as follows:

Step 1: Choose the substitution u in terms of x . How do we choose the right substitution? The rule of thumb which began to emerge in the preceding set of examples is this: we look for a part of the integrand which has the form $g(f(x))$ (such as $(x^2 + 1)^2$ in [Example 3](#), or $\sin \sqrt{x}$ in [Example 6](#)) and set $u = f(x)$. Another way of putting this is as follows: when looking at an integrand containing, say, $\sin \sqrt{x}$, we ask ourselves: “*what simplification might turn this expression into something we can integrate easily?*” In the case in point, if $\sin \sqrt{x}$ were to be simplified to $\sin u$, our task would become much easier. This suggests that we might try a substitution of $u = \sqrt{x}$.

Note

A useful general principle in approaching integration by substitution is to look at our integrand, which may comprise a product of two functions, and try to spot a “function-derivative pair”. If we are able to use substitution, we would expect to find the “function” component bracketed inside another function, and the derivative component present in the other factor of the product, with or without an accompanying constant coefficient.

Step 2: Obtain an expression for dx . This could be described, using an older term from the teaching of calculus, as “calculating the differential” – this being the name given to a symbol such as dx on its own. We achieve this by applying the “formula”:

$$du = \frac{du}{dx} dx$$

differentiating our substitution expression for u with respect to x , and rearranging to obtain an expression for dx . This will usually be an expression in terms of both x and du .

(In my earlier comments, I have been scathing about the mathematical rigour of taking what we are taught to regard as the *symbol* du/dx apart and treating it like a fraction. I might point out in mitigation, however, that we are also taught a number of rules for manipulating derivatives in Leibniz notation which could lead us to believe that treating du/dx like a

fraction in some limited circumstances is not wholly wrong-headed. I have persevered with the treatment of du/dx as if it were a fraction since this approach allows us to illustrate in a shorthand form the work necessary to change our integrand into a form where we can reliably integrate with respect to the substitution variable u .)

Step 3: Make the substitution. This involves making substitutions in the original integral using both the chosen substitution formula for u in terms of x , and the formula for the “differential” dx in terms of x and du . After making both of these substitutions – and the method requires both, not just one of them – we should find that all instances of x in the integral have been removed – perhaps by cancellation, or because the integrand already contains part of the formula we obtained for dx . (We have seen this latter behavior on several occasions in the above examples.) If instances of x still remain after these substitutions, then either we have chosen the wrong substitution, and we should return to Step 1 and think again, or the integral cannot actually be solved using the substitution method.

Step 4: Integrate the resulting integral with respect to u . As we have seen in the examples, this can be a very simple matter if the problem we start with is particularly suited to the substitution approach.

Step 5: Return the expression to the original variable x . Use the formula for u in terms of x to expand out the result of the integration in Step 4 to arrive at a final expression in x . Don’t forget that it may be useful to tidy the final expression up by factorising out any common factors, or making any other rearrangements which might lead to greater conciseness. Such rearrangements will often cause the result to appear in a broadly similar form to that of the original integrand.

I have reproduced this algorithm as a list of five simple steps, without my explanatory notes, in [Appendix 1 Section 1](#).

5 When Does Substitution Work?

We are now in a position to give at least a preliminary answer to the question of what kinds of function can best be integrated using the method of substitution.

As a preparatory step, let’s review what we are doing when we integrate by substitution. Recalling the Chain Rule for differentiation which we used as a starting point for our discussion, we can say that in using substitution, we transform a “difficult” integral into a simpler one by choosing a formula for a new variable u in terms of x ; after some manipulation (in particular, obtaining a formula for dx by differentiating our formula for u), we are left with a simpler integral, hopefully comprising one simple function in terms of u which we can integrate with respect to u - or possibly a sum of such functions. The method of substitution could be written in a shorthand form by the following equation:

$$\int g'(f(x))f'(x)dx = \int g'(u)du, \text{ where } u = f(x)$$

We move from the use of dx to the use of du by first noting the following equivalence, which arises from our formula for the substitution variable u :

Integration by Substitution

$$\frac{du}{dx} = f'(x)$$

This allows us to make a notational rearrangement in order to obtain the following expression for dx :

$$dx = \frac{du}{f'(x)}$$

The above expression equating the original integral to a new integral in terms of u can now be explained more fully using an intermediate step. This step is made by substituting into the original integral our expressions for both u and dx . The full details are as follows:

$$\int g'(f(x))f'(x)dx = \int g'(u)f'(x)\frac{du}{f'(x)} = \int g'(u)du$$

If we denote the “antiderivative” of $g'(u)$ by $g(u)$, we can then make the general statement that the integral we wish to evaluate will be equal to:

$$g(u) + C = g(f(x)) + C$$

This abstract and symbolic reasoning would actually allow us to write down almost immediately the following solution to any integral of the type we have been considering:

$$\int g'(f(x))f'(x)dx = g(f(x)) + C$$

which is, of course, nothing more than the Chain Rule for differentiation “in reverse” – that is, a function of a function expressed as the integral of the expression for its derivative which the Chain Rule provides.

Let’s take a look at how some of the integrands in our preceding examples can be decomposed in the light of the general form set out above.

Example 1

$$g'(f(x)) = 1 + \sin x + \sin^2 x; g'(x) = 1 + x + x^2$$

$$f(x) = \sin x; f'(x) = \cos x$$

$$g(x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3; g(f(x)) = \sin x + \frac{1}{2}\sin^2 x + \frac{1}{3}\sin^3 x$$

Example 2

$$g'(f(x)) = (\sin x - \cos x)^2; g'(x) = x^2$$

$$f(x) = \sin x - \cos x; f'(x) = \sin x + \cos x$$

Integration by Substitution

$$g(x) = \frac{1}{3}x^3; g(f(x)) = \frac{1}{3}(\sin x - \cos x)^3$$

Example 3

$$g'(f(x)) = (x^2 + 1)^2; g'(x) = x^2$$

$$f(x) = x^2 + 1; f'(x) = 2x \text{ so that } \frac{1}{2}f'(x) = x$$

Our integrand was therefore of the following form:

$$\frac{1}{2}g'(f(x))f'(x)$$

To complete the decomposition:

$$g(x) = \frac{1}{3}x^3; g(f(x)) = \frac{1}{3}(x^2 + 1)^3$$

However, due to the factor of $1/2$ which appeared in our explanation above, our final solution was found to be:

$$\frac{1}{6}(x^2 + 1)^3 + C$$

(We notice that in this case, $f'(x)$, which we noted to be $2x$, contains a coefficient of 2, which is not present in our integrand. We can, however, note that our integrand does not actually contain $f'(x)$ itself, but only $(1/2)f'(x)$. This coefficient of $1/2$ can be brought outside the integral sign, upon which the remaining integrand is of the exact form we require to apply our technique. After integration, we can apply a corrective coefficient of $1/2$ to the result. This can be considered an alternative explanation to the reasoning set out in the commentary on the example itself.)

From these decompositions, and in the light of the working shown in the examples themselves, we can conclude that integrands expressed in the form $g'(f(x))f'(x)$ can generally be integrated by substitution. [Example 3](#), as well as our knowledge of simple integration, tells us in addition that the presence of a constant coefficient in the expression which would occupy the place of $f'(x)$ (if, for example, the given integrand contains not exactly $f'(x)$ but a fraction of $f'(x)$) does not prevent the use of substitution, since a corrective coefficient can be applied to the result of our efforts.

We can finally conclude, then, that integrands comprising the product of two simpler functions with or without a constant coefficient, which have the form:

$$kg'(f(x))f'(x)$$

can usually be integrated by substitution.

(In the light of this revised general formula, we might extend the decomposition of the integrand in [Example 3](#) to include an additional term of $k = 1/2$.)

Example 6

We might think at first that it would be hard to fit [Section 3 Example 6](#) above into our structure. This difficulty should evaporate, however, once we rewrite the integral in the equivalent, but in this case more informative, form:

$$I = \int x^{-1/2} \sin x^{1/2} dx$$

This allows us to give the following decomposition for [Example 6](#):

$$g'(f(x)) = \sin x^{1/2}; \quad g'(x) = \sin x$$

$$f(x) = x^{1/2}; \quad f'(x) = \frac{1}{2}x^{-1/2} \text{ so that } 2f'(x) = x^{-1/2}$$

Our integrand therefore had this form:

$$2g'(f(x))f'(x)$$

Even though the factor of 2 was not visible in the original integrand- and we can complete the decomposition as follows:

$$g(x) = -\cos x; \quad g(f(x)) = -\cos x^{1/2}$$

The factor of 2 in the expression representing the form of the integrand, however, gave us this solution:

$$-2 \cos x^{1/2} + C$$

From all this reasoning, we can see that the method of integration by substitution can be regarded as a kind of Chain Rule in reverse. Whereas in order to use the Chain Rule we must observe that the function to be differentiated is of the form $g(f(x))$ - a “function of a function” - in order to use integration by substitution, we must first spot that our integrand is a product of two simpler functions (with or without a constant coefficient), and that the two functions are related by differentiation so that the integrand has the form $kg'(f(x))f'(x)$. The presence of the constant k in this expression covers situations where our integrand contains not $f'(x)$ itself, but $kf'(x)$ - we noted such a case in our discussion of [Example 3](#) above, where k was found to have the value $1/2$.

[Section 3 Example 5](#) above and its accompanying note tell us that we can also expect to be able to integrate functions of the form $f(ax + b)$ using the substitution $u = ax + b$. At first sight, such functions might seem not to conform to the structure we observed in the examples cited earlier in this section. However, we can decompose the integrand in [Section 3 Example 4](#) as follows:

Integration by Substitution

$$g'(f(x)) = \cos(2x + 3); g'(x) = \cos x$$

$$f(x) = 2x + 3; f'(x) = 2 \text{ so that } \frac{1}{2}f'(x) = 1$$

Our integrand therefore needs to be expressed with the help of a “hidden” constant factor – indeed, it is of the form:

$$\frac{1}{2}g'(f(x))f'(x)$$

so that $k = 1/2$, and we can complete the decomposition as follows:

$$g(x) = \sin x; g(f(x)) = \sin(2x + 3)$$

Taking into account our discovery of the need for a constant coefficient k in making our integrand, fit into the standard form for substitution, we find that our final solution is actually:

$$\frac{1}{2}\sin(2x + 3) + C$$

This decomposition points the way to regarding our integrand as falling into the structure we have set out for the use of substitution - it's just that one of the two functions in the product we are looking for is a constant function – in this case, one half of that constant function. As we have remarked before, our original integrand lacks this constant coefficient – but we can express its ghostly presence using the factor k in our standard formula for the purposes of qualifying it for integration by substitution and apply a corrective coefficient to our result. We can express this approach in symbolic terms by providing the following slightly more general version of the result of integrating by substitution:

$$\int kg'(f(x))f'(x)dx = kg(f(x)) + C$$

The purpose of this explanation is to demonstrate that functions of the form $f(ax + b)$ are simply special cases of the general pattern $kg'(f(x))f'(x)$, where (renaming our function as $F(ax + b)$ to avoid confusion):

$$g'(f(x)) = F(ax + b); f(x) = ax + b; f'(x) = a; k = \frac{1}{a}$$

As a final remark on the subject of functions susceptible to integration by substitution, we should note that it is also possible to integrate *sums* of functions of the form $kg'(f(x))f'(x)$, by treating each term as a separate integral and then adding the results. If necessary, this might involve using different substitutions for different terms. For example, the integral:

$$I = \int \left(e^{2x} + \frac{x}{\sqrt{1-4x^2}} \right) dx = \int e^{2x} dx + \int \frac{x}{\sqrt{1-4x^2}} dx$$

can be treated as two separate integrals; the first one needs the substitution $u = 2x$, while the second can be solved using $u = 1 - 4x^2$.

6 Definite Integrals Using Substitution

Before proceeding to explore some more sophisticated uses of the substitution method, it would be useful to remark briefly on a minor but possibly very useful consequence of integrating by substitution in situations where we have to evaluate a definite integral – so far, to illustrate and explain the method, we have only considered indefinite integrals.

Let us therefore turn our minds to the evaluation of the following definite integral:

$$I = \int_{-1}^1 x^2(x^3 + 1)^4 dx$$

Treating the problem first as an indefinite integral, our earlier reasoning should lead us to try a substitution of $u = x^3 + 1$. This gives us:

$$\frac{du}{dx} = 3x^2$$

so that:

$$dx = \frac{1}{3x^2} du$$

Proceeding to make these substitutions, we find that the x^2 in the denominator of the expression for dx cancels out with the x^2 factor in the integrand, so that we are left with the following integral in terms of u :

$$\int \frac{1}{3} u^4 du$$

which integrates simply to:

$$\frac{1}{15} u^5 + C = \frac{1}{15} (x^3 + 1)^5 + C$$

The final solution to our definite integral would then be given by:

$$\left[\frac{1}{15} (x^3 + 1)^5 \right]_{-1}^1$$

which would require us to substitute the values 1 and -1 into the expression in x contained inside the square brackets.

However, since we have arrived at this solution using substitution, there is an alternative which may frequently provide a less laborious route to the end result. In fact, it is not

necessary to take the final step of a return to an expression in x if we can find the values of u which correspond to the limits of the definite integral in x . We can do this by substituting each of the limits of integration into the substitution formula we have chosen, as follows:

$$x = -1 \text{ gives } u = (-1)^3 + 1 = 0$$

$$x = 1 \text{ gives } u = 1^3 + 1 = 2$$

These results allow us to express the solution in terms of u , as follows:

$$I = \left[\frac{1}{15} u^5 \right]_0^2 = \frac{32}{15}$$

Since the aim of integration by substitution is to put ourselves into a position where we are integrating a rather *simpler* function in terms of u , we might well expect that avoiding a return to an expression in terms of x before evaluating the definite integral would normally save us at least some effort in calculation.

7 Some Standard Rules for Integration

We are now in a position to extract from our experience of integration by substitution some useful standard rules for integrating functions of particular types: these rules will make use of the principles of integration by substitution which we have been gradually piecing together in the preceding sections.

First of all, we must recall an identity which was stated towards the end of [Section 5](#) above, where we discussed the question of why the substitution technique works, and its relationship with the Chain Rule for differentiation. This identity could be regarded as a statement of the Chain Rule “in reverse”, so that it can be applied to integration; it was as follows:

$$\int g'(f(x))f'(x)dx = g(f(x)) + C$$

We can use this fundamental formula, together with the technique of integration by substitution itself, to derive some useful standard rules for integration which will allow us to write down directly the integrals of some functions which might otherwise require detailed work.

Two Useful Rules

In order to make a start, let's consider the above formula with a fairly simple function in place of $g'(x)$ in the integrand: we will set $g'(x) = x^n$. This means we need to obtain a formula for the general integral:

$$I = \int (f(x))^n f'(x) dx$$

We will walk through the process of integration by substitution here (since the integrand is of the exact form we need to apply this technique). We will substitute $u = f(x)$. By differentiation, this gives us the following (again, a quite general result):

Integration by Substitution

$$\frac{du}{dx} = f'(x) \text{ so that } dx = \frac{du}{f'(x)}$$

Putting our substitution and the expression for dx into the original integral, we have the following in terms of u :

$$\int u^n f'(x) \frac{du}{f'(x)} = \int u^n du$$

(The right-hand side is obtained by cancelling out the occurrences of $f'(x)$.)

This integrates very simply – by the standard formula for the integration of a simple power of our chosen variable – to give the result:

$$\frac{1}{n+1} u^{n+1} + C$$

and reversing the substitution then gives us the following final result:

$$I = \frac{1}{n+1} (f(x))^{n+1} + C$$

We can now state the first of our new rules for integration. It runs as follows:

$$\int (f(x))^n f'(x) dx = \frac{1}{n+1} (f(x))^{n+1} + C$$

We can expect this to work for any suitable function $f(x)$. Let's present a few examples:

Example 1

$$\int \sin^3 x \cos x dx = \frac{1}{4} \sin^4 x + C$$

because

$$f(x) = \sin x, f'(x) = \cos x \text{ and } n = 3$$

Example 2

$$\int 3x^2 (x^3 - 5)^4 dx = \frac{1}{5} (x^3 - 5)^5 + C$$

because

$$f(x) = x^3 - 5, f'(x) = 3x^2 \text{ and } n = 4$$

Example 3

$$\int x^3(x^4 + 2)^2 dx = \frac{1}{12}(x^4 + 2)^3 + C$$

This seems a slightly less immediate application of our new rule. To see why it works, however, we need only rewrite the left-hand side integral as follows:

$$\frac{1}{4} \int 4x^3(x^4 + 2)^2 dx$$

And then note that our rule directly applies to the revised integral to give:

$$\int 4x^3(x^4 + 2)^2 dx = \frac{1}{3}(x^4 + 2)^3 + C$$

because

$$f(x) = x^4 + 2, f'(x) = 4x^3 \text{ and } n = 2$$

(In other words, by rewriting the integral with a coefficient taken outside the integral sign, we can now see clearly the function and its derivative which we require to apply the rule.)

Our initial solution then arises by simply re-applying the coefficient of $1/4$ to the result obtained from our rule.

One issue which arises from this first useful rule, however, is the question of what happens when we are faced with a situation where $n = -1$. In such a case, the right-hand side expression for the solution to our integral would not make any sense, since the fractional coefficient it contains would be undefined in this situation. It will therefore be useful to investigate whether we can derive an alternative rule to deal with this particular situation.

We should note first that we can rewrite our integral where $n = -1$ as follows:

$$\int (f(x))^{-1} f'(x) dx = \int \frac{f'(x)}{f(x)} dx$$

As before, we will substitute $u = f(x)$, and this gives us the same result for dx as we obtained before, that is:

$$\frac{du}{dx} = f'(x) \text{ so that } dx = \frac{du}{f'(x)}$$

Now we can put everything back into our original integral and arrive at the following integral in terms of u :

Integration by Substitution

$$\int \frac{f'(x)}{u} \cdot \frac{du}{f'(x)} = \int \frac{1}{u} du$$

(Once again, we see that the two instances of $f'(x)$ cancel each other out.)

This is now a standard integral, whose solution is:

$$\ln|u| + C$$

and by reversing the substitution we will arrive at the following solution in terms of x :

$$I = \ln|f(x)| + C$$

We have now covered the situation where our first rule falls down – it can only be relied upon in cases where $n \neq -1$ - and state this additional rule of integration:

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

As before, it will be helpful to look at a few examples of this rule in action.

Example 4

$$\int \frac{\cos x}{\sin x} dx = \ln|\sin x| + C$$

because

$$f(x) = \sin x \text{ and } f'(x) = \cos x$$

We should note that the integrand we have considered here is usually rewritten as $\cot x$, the trigonometric cotangent function. This function is defined as follows:

$$\cot x = \frac{1}{\tan x}$$

Noting the common trigonometric identity for $\tan x$:

$$\tan x = \frac{\sin x}{\cos x}$$

we can conclude that:

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

That said, our example integral could be rewritten as the following standard integral:

Integration by Substitution

$$\int \cot x \, dx = \ln|\sin x| + C$$

Example 5

$$\int \frac{2x}{x^2 + 1} dx = \ln(x^2 + 1) + C$$

because

$$f(x) = x^2 + 1 \text{ and } f'(x) = 2x$$

Example 6

$$\int \frac{x^2}{x^3 - 3} dx = \frac{1}{3} \ln|x^3 - 3| + C$$

Like [Example 3](#) above, this entails the use of a constant coefficient. We can rewrite our integral as:

$$\frac{1}{3} \int \frac{3x^2}{x^3 - 3} dx$$

and observe that our rule gives us:

$$\int \frac{3x^2}{x^3 - 3} dx = \ln|x^3 - 3| + C$$

because

$$f(x) = x^3 - 3 \text{ and } f'(x) = 3x^2$$

Re-application of the coefficient of $1/3$ gives us our desired result.

Some Useful Consequences

We can make these two rules for integration even more useful by looking at how they apply to one particular function:

$$f(x) = ax + b$$

This function is the general linear expression in x . We immediately note the consequence regarding its derivative – $f'(x) = a$.

Putting this simple function and its derivative into the first of our two rules (the rule which applies when $n \neq -1$), we have the following identity:

Integration by Substitution

$$\int a(ax + b)^n dx = \frac{1}{n+1} (ax + b)^{n+1} + C$$

If we multiply both sides by a coefficient of $1/a$, we will obtain the following consequence:

$$\frac{1}{a} \int a(ax + b)^n dx = \frac{1}{a(n+1)} (ax + b)^{n+1} + C$$

Now, we should notice here that in the integral on the left-hand side, our coefficient of $1/a$ and the coefficient of a inside the integral sign (which can, of course, be brought outside the integral sign!) cancel each other out. This leaves us with the following useful rule for integrating linear expressions raised to a power:

$$\int (ax + b)^n dx = \frac{1}{a(n+1)} (ax + b)^{n+1} + C$$

In situations where $a = 1$ (and I will rename b as k to represent the constant term inside the integrand) this reduces to the following:

$$\int (x + k)^n dx = \frac{1}{n+1} (x + k)^{n+1} + C$$

These useful rules allow us to write down directly some integrals which we might not have been able to deal with so quickly before. Here are a couple of examples:

$$\int (2x + 5)^3 dx = \frac{1}{8} (2x + 5)^4 + C$$

$$\int (x - 2)^4 dx = \frac{1}{5} (x - 2)^5 + C$$

A similar useful result can be obtained from the second general rule we established in the case where $n = -1$. Once again, we will start by setting:

$$f(x) = ax + b \text{ so that } f'(x) = a$$

Our second rule gives us this identity:

$$\int \frac{a}{ax + b} dx = \ln|ax + b| + C$$

Multiplying both sides by $1/a$ (and this time, we will cancel out the a in the integrand's numerator with our factor of $1/a$ straight away) we obtain the following useful rule:

$$\int \frac{1}{ax + b} dx = \frac{1}{a} \ln|ax + b| + C$$

Finally, setting $a = 1$ and renaming b as k the way we did a little earlier gives us the

Integration by Substitution

following rule for simpler linear expressions:

$$\int \frac{1}{x+k} dx = \ln|x+k| + C$$

We can additionally use the substitution technique to derive two useful rules which can be applied when integrating expressions containing functions of any kind at all. Carrying on from the last result, let's consider the following very general integral, where k represents some constant value:

$$\int f(x+k) dx$$

We will further adopt the convention that the integral of the (unknown) function $f(x)$ in the above integral is denoted by $F(x) + C$. Written out as an equation, this would be:

$$\int f(x) dx = F(x) + C$$

With this convention in mind, let's look back at the integral of $f(x+k)$ I set out earlier. We should expect to be able to approach the solution of such an integral by making the substitution:

$$u = x + k \text{ so that } \frac{du}{dx} = 1 \text{ and so } dx = du$$

We then have an integral in terms of u as follows:

$$\int f(u) du$$

By the convention we adopted to represent the integral of $f(x)$, this has the solution:

$$F(u) + C$$

and, reversing the substitution, we can then state the following general rule of integration:

$$\int f(x+k) dx = F(x+k) + C$$

Let's see an example of this rule in action:

Example 7

$$\int \cos(x+3) dx = \sin(x+3) + C$$

In this example, our constant $k = 3$, and we were able to make use of the standard integral:

$$\int \cos x \, dx = \sin x + C$$

Our final rule is of a similar type, and is once again derived using substitution on an integral expressed symbolically, with the same conventions as those we used for the preceding rule. We will consider:

$$\int f(kx) \, dx$$

Here, we will substitute:

$$u = kx \text{ so that } \frac{du}{dx} = k \text{ and so } dx = \frac{1}{k} du$$

This gives us the following integral in terms of u :

$$\int \frac{1}{k} f(u) \, du = \frac{1}{k} \int f(u) \, du$$

The integral on the right-hand side, of course, has the solution:

$$F(u) + C$$

so that we can reverse our substitution and apply the coefficient of $1/k$ which we took outside the integral sign to write down our solution as follows:

$$\int f(kx) \, dx = \frac{1}{k} F(kx) + C$$

As before, an example will serve to show the kind of result we can get from this rule.

Example 8

$$\int \cos 2x \, dx = \frac{1}{2} \sin 2x + C$$

We used the same standard integral as was applied in [Example 7](#) - the integral of $\cos x$ - to obtain this fairly familiar result, where our constant $k = 2$.

We have here a number of useful rules for integration which will allow us to write down immediately solutions to integrals which would previously have required walking through the substitution technique. I have listed all of these rules for ease of reference in [Appendix 4 Section 2](#).

8 Algebraic Substitutions

You are likely to be already familiar with the idea that some integrals which look difficult can be made more tractable by performing some simple algebra on the integrand, in order to rearrange it into a form which is easier to deal with. Indeed, we referred to this possibility when considering [Example 3 in Section 3](#) above.

We will now proceed to examine some examples of the use of our substitution technique where the substitution is chosen so as to allow us to carry out some simple algebra on our integrand which will yield up a more easily integrated expression. Once again, we will do this by exhibiting some examples and accompanying them with plenty of explanation as we go.

Example 1

Here we will consider the following indefinite integral:

$$I = \int x(x+1)^3 dx$$

We might first remark that the integrand does not fit the desired structure for integration by substitution as we set it out in [Section 4](#) above. However, let's persevere in this case, and see how far we can get by trying substitution anyway. Trying to apply our algorithm as best we can, we note that the most complicated part of the integrand, comprising a compound function, is the cubed factor in brackets. We might therefore suggest a substitution of $u = x + 1$, since we know how to integrate u^3 .

We should immediately note that this simple substitution gives us a very simple relationship between dx and du . This is because:

$$\frac{du}{dx} = 1 \text{ so that } dx = du$$

This only leaves us with the question of what should be done with the x term in the integrand, since unlike in our earlier examples, we do not have anything provided in our expression for dx which we can use for cancellation. The answer comes from remembering that here, we are using substitution to help us do some algebraic rearrangement before integrating; making use of the simple substitution expression we have chosen, we should be led to the conclusion that the way forward is simply to write x in terms of u using a rearrangement of our substitution formula, giving us $x = u - 1$. Our integral now becomes:

$$\int (u-1)u^3 du = \int (u^4 - u^3) du$$

This is a simple difference of powers of u , which we will integrate with ease in a moment. First, however, let's think briefly about what we have done here: we have chosen a substitution which, despite the integrand not looking quite suitable for this technique, has allowed us to transform the cube of a bracketed expression in x to create a simple power of u . Because the substitution was an uncomplicated one, we have been able to expand the x term

Integration by Substitution

into a simple bracketed expression in u - not raised to a power, but instead amenable to straightforward “multiplying out”, to give us an easy integral to evaluate.

Let’s complete the job. Our integral in u integrates using standard rules to give:

$$\frac{1}{5}u^5 - \frac{1}{4}u^4 + C = u^4 \left(\frac{1}{5}u - \frac{1}{4} \right) + C$$

Finally, by reversing the substitution, we obtain the following solution in x :

$$(x+1)^4 \left(\frac{1}{5}(x+1) - \frac{1}{4} \right) + C = \frac{1}{5}(x+1)^4 \left(x - \frac{1}{4} \right) + C$$

The final step here involves expanding the right-hand bracketed term in the first expression above as follows:

$$\frac{1}{5}(x+1) - \frac{1}{4} = \frac{1}{5}x + \frac{1}{5} - \frac{1}{4} = \frac{1}{5}x - \frac{1}{20} = \frac{1}{5} \left(x - \frac{1}{4} \right)$$

Example 2

In this example, we’ll attempt to generalise a little from the previous one, in order to exhibit a new class of functions which can be integrated by substitution, using the “algebraic” approach we saw in the previous example. We therefore set ourselves the task of integrating:

$$I = \int (x+2)(x+3)^4 dx$$

Noting that this is only slightly more complicated than [Example 1](#), we can immediately proceed to suggest a substitution of $u = x + 3$, which as before will have as consequence the very simple relationship $dx = du$. Furthermore, we note that the first factor in the integrand can be expressed easily in terms of u . Rearranging our substitution formula, we have:

$$x = u - 3 \text{ so that } x + 2 = u - 1$$

On substitution, therefore, we obtain the integral:

$$\int (u-1)u^4 dx = \int (u^5 - u^4) dx$$

This doesn’t look an awful lot different from the integral in [Example 1](#), and integrates to:

$$\frac{1}{6}u^6 - \frac{1}{5}u^5 + C = u^5 \left(\frac{1}{6}u - \frac{1}{5} \right) + C$$

Finally, reversing the substitution gives us:

$$(x+3)^5 \left(\frac{1}{6}(x+3) - \frac{1}{5} \right) + C = \frac{1}{2}(x+3)^5 \left(\frac{1}{3}x + \frac{3}{5} \right)$$

The final step results from the same kind of simple algebra we made use of at the end of [Example 1](#). The detailed steps we perform on the right-hand bracketed term in the first expression run as follows:

$$\frac{1}{6}(x+3) - \frac{1}{5} = \frac{1}{6}x + \frac{1}{2} - \frac{1}{5} = \frac{1}{6}x + \frac{3}{10} = \frac{1}{2}\left(\frac{1}{3}x + \frac{3}{5}\right)$$

It should be clear that we can use exactly the same technique, with one of two of the numeric constants altered as necessary, to solve any problem of the form:

$$I = \int (x+a)(x+b)^n dx$$

by choosing the substitution $u = x + b$. A demonstration of the integration of this general expression is given in [Appendix 5 Section 2](#).

It is worth remarking additionally that it doesn't affect the usefulness of our method if the power of $x + b$ is negative, so that we can also use the method of this example to solve problems of the form:

$$I = \int \frac{x+a}{(x+b)^n} dx$$

Example 3 - Substitution with a Difference

This example will look somewhat familiar after the preceding two, but we will approach it in a different way. This will serve to illustrate a use of the substitution technique which is sometimes encountered, and is generally adopted in order to reduce the complexity of the integral to be evaluated after the substitution has been made.

Our problem will be to evaluate the indefinite integral:

$$I = \int (x+1)\sqrt{x+2} dx$$

Apart from the fact that the second factor in the product which comprises the integrand is raised to a fractional power, namely $1/2$, this integral clearly matches the structure of the preceding two examples. We might therefore expect to be able to use the substitution $u = x + 2$. This does, however, require us to integrate an expression in u with fractional powers, which, though entirely feasible, we might feel to be algebraically vexatious and, worse, prone to error.

We can therefore ask ourselves: is there any substitution we might perform which would make the fractional power, ie the square root, disappear from view so that we are left with whole-number powers only? And indeed there is (though it may not immediately pop into the mind of itself): if we express our substitution in a reversed form, ie as an expression for x in terms of u , we will find that the substitution $x = u^2 - 2$ will transform the square root factor in our integrand into the simple value u .

Integration by Substitution

How do we arrive at this choice of substitution? Well, we want to end up with the result that:

$$u = \sqrt{x + 2}$$

Simple algebra then gives:

$$u^2 = x + 2 \text{ so that } x = u^2 - 2$$

(We should pause a moment here to remark that there is, of course, nothing wrong with expressing our substitution as a simple expression for u , as we did a few lines ago when I explained how we might get to the substitution formula I originally quoted. However, I have decided to show the substitution in the form I have because this is a way you will see substitutions described, and it is useful to get a little experience of it!)

The first factor will be transformed into:

$$u^2 - 2 + 1 = u^2 - 1$$

We are now just left with the question of finding an expression for dx . Since our substitution this time is given in terms of u , we might try differentiating the expression for x with respect to u . as follows:

$$\frac{dx}{du} = 2u \text{ so that } dx = 2u du$$

After substitution, our integral now becomes:

$$\int (u^2 - 1)u \cdot 2u du = \int 2u^2(u^2 - 1)du = \int (2u^4 - 2u^2)du$$

This is now easily integrated (without any fractional powers) to give us, in terms of u :

$$\frac{2}{5}u^5 - \frac{2}{3}u^3 + C = 2u^3 \left(\frac{1}{5}u^2 - \frac{1}{3} \right) + C$$

Now, however, we come to the final matter – that of reversing the substitution. As initially set out, the expression for x in terms of u is fairly unhelpful as it stands: there is nothing else for it but to rearrange it to give an expression for u which we can then put back into the above solution in u . Simple algebra gives us:

$$u^2 = x + 2, \text{ so that } u = \sqrt{x + 2}$$

and we can now reverse the substitution in the manner we have adopted every time so far, to obtain:

$$2(x + 2)^{3/2} \left(\frac{1}{5}(x + 2) - \frac{1}{3} \right) + C = \frac{2}{5} \sqrt{(x + 2)^3} \left(x + \frac{1}{3} \right) + C$$

The final step takes place by carrying out the following work on the right-hand bracketed term in the first expression:

$$\frac{1}{5}(x+2) - \frac{1}{3} = \frac{1}{5}x + \frac{2}{5} - \frac{1}{3} = \frac{1}{5}x - \frac{1}{15} = \frac{1}{5}\left(x + \frac{1}{3}\right)$$

We note that one fractional power does reappear at the very end of the process, but not in a manner which is likely to cause us any alarm.

We may now point out that our initial discussion of this example, where we noted that it should respond to the substitution $u = x + 2$, as long as we are not deterred by having to deal with fractional powers of u , leads us to the conclusion that this type of substitution will handle any problem of the form:

$$I = \int (x+a)(x+b)^p dx$$

where p is any positive rational number, ie a number greater than zero with or without a fractional part.

Example 4: A More General Case

In order to go a little further in our survey of the types of integral which this technique will help us to tackle, let's take a look at a substitution-based approach to the following integral:

$$I = \int \frac{3x+5}{(2x+3)^3} dx$$

In overall appearance, this is not wholly dissimilar to the type of integral we have been concerned with previously in this section, except that one linear factor is raised to a negative power. We will therefore start off by rewriting the integral for convenience in the following form:

$$I = \int (3x+5)(2x+3)^{-3} dx$$

and immediately declaring, in line with the earlier examples in this section, that we will try using the substitution $u = 2x + 3$. Now, this is a substitution of the form $u = ax + b$, a type we remarked on in the note following [Section 3 Example 5](#) above. In accordance with those remarks, we immediately note that our chosen substitution gives us this equivalence:

$$dx = \frac{1}{2} du$$

We should just pause for one moment as we consider how the first factor in the product which constitutes our integrand can be converted into an expression in terms of u . If we first make x the subject of our substitution expression, obtaining the following:

$$x = \frac{1}{2}(u-3)$$

we can substitute this into our first factor, obtaining:

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$$\frac{3}{2}u - \frac{9}{2} + 5 = \frac{3}{2}u + \frac{1}{2} = \frac{1}{2}(3u + 1)$$

Making all the substitutions we have now arrived at, our integral is transformed into:

$$\int \frac{1}{2} \left(\frac{1}{2}(3u + 1) \right) u^{-3} du = \frac{1}{4} \int (3u^{-2} + u^{-3}) du$$

Integrating term by term yields the result:

$$\frac{1}{4} \left(-3u^{-1} - \frac{1}{2}u^{-2} \right) + C$$

We can now proceed to finish the job by reversing the substitution. The result is transformed as follows:

$$-\frac{1}{4} \left(\frac{3}{2x+3} + \frac{1}{2(2x+3)^2} \right) + C$$

We can combine the two fractions inside the brackets into one by the following bit of algebra, where we multiply the numerator and denominator of the first fraction by $2(2x+3)$:

$$-\frac{1}{4} \left(\frac{6(2x+3)}{2(2x+3)^2} + \frac{1}{2(2x+3)^2} \right) + C = -\frac{6(2x+3) + 1}{8(2x+3)^2} + C$$

Let's take things one step further by multiplying out and simplifying the numerator, so that we arrive at the following:

$$-\frac{12x+19}{8(2x+3)^2} + C$$

which brings the solution into a similar shape to that of the integrand we started with.

Now we are in a position to go a little further with our generalisation and say that we can use integration by substitution to solve any integral of the form:

$$I = \int (ax+b)(cx+d)^p dx$$

where p can be either positive or negative, and I give a general solution to any such integral using symbols instead of actual values for the constants in [Appendix 5 Section 3](#).

Note

To help in deciding on a substitution formula which is to be expressed in the “reversed” form we demonstrated in [Example 3](#) above, we might offer the following few comments: first, the objective is to use the substitution to “hide” a particularly awkward part of the integrand in such a way that it does not hopelessly overcomplicate the rest of it; second, it may take

several attempts before a really useful one is found; and third, our choice of substitution should allow us to simplify the integrand algebraically in order to get to something we can handle more easily.

Some algebraic justification for the choice we made is given in the explanation of [Example 3](#) above, and it can also be useful to start by setting a portion of the integrand equal to the substitution variable u , as we did in [Example 3](#), and then rearranging this equation to express the substitution in whatever form we may desire. (We could, of course, equally well stop after the first step – one reason for making a rearrangement might, however, be to make the differentiation of the substitution formula more straightforward.)

When we encounter in mathematical literature the glib statement of one of these “reversed” substitutions, it’s worth bearing in mind that any glibness in the statement can only be possible with the benefit of hindsight (or extensive experience). It’s also the case that the substitution quoted will, of course, be the one that works, and it’s often difficult, or deemed a waste of space, to set out the process of trial and error which finally yielded it up.

Example 5: A Horse of a Different Colour

Before leaving this consideration of algebraic substitution, let’s take a look at a somewhat different kind of function which can be integrated using an approach broadly similar to the one which has worked in the previous three examples.

We are going to consider the following integral:

$$I = \int \frac{x^3}{x^2 + 1} dx$$

Now, although we might be inclined to think that because x^3 differentiates to give $3x^2$, this integrand might contain a “function-derivative pair” of the kind that allows us to fit it into the standard form for integration by substitution which we explained in [Section 4](#) above, we will find if we press this attempt any further that the integrand is *not* in fact in this form at all!

However, it can be tackled using a relatively simple substitution. We must first bear in mind that our aim in using algebraic substitution is to transform our integral into a *simpler* form than its original form. In the case of rational functions such as the current integrand, this might involve radically simplifying the denominator in order to turn the integral into a much simpler sum of powers, which can be dealt with term by term.

With this in mind, then, we might start our work here by trying a substitution of $u = x^2 + 1$ - in other words, we will try substituting the entire denominator with our new variable u .

This leads in the first instance to the following consequence:

$$\frac{du}{dx} = 2x, \text{ so that } dx = \frac{1}{2x} du$$

If we now insert these facts into the original integrand, it will change to the following:

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$$\int \frac{x^3}{2ux} du$$

The outstanding x term in the denominator can be cancelled with the numerator, which will leave a numerator of x^2 . A rearrangement of our substitution formula will give us the equivalence $x^2 = u - 1$. If we also take the constant factor of $1/2$ outside the integral sign, we will be left with the following integral in terms of u :

$$\frac{1}{2} \int \frac{u-1}{u} du = \frac{1}{2} \int \left(1 - \frac{1}{u}\right) du$$

which is fairly easy to integrate – indeed, we are left with the following solution in terms of u :

$$\frac{1}{2}(u - \ln|u|) + C$$

Taking the final step of reversing the substitution, here is our solution in terms of x :

$$\frac{1}{2}(x^2 + 1 - \ln(x^2 + 1)) + C$$

We can safely drop the modulus brackets following the logarithmic function here, since it will be impossible for $x^2 + 1$ to take on a negative value!

One thing which is noticeable about this solution, however, is that it contains an explicit constant term: if we take the constant term 1 outside the brackets (not forgetting to multiply it by $1/2$), then our solution looks like this:

$$\frac{1}{2}(x^2 - \ln(x^2 + 1)) + \frac{1}{2} + C$$

We could choose to rewrite this as follows:

$$\frac{1}{2}(x^2 - \ln(x^2 + 1)) + C' \text{ where } C' = C + \frac{1}{2}$$

Now, this is every bit as good a solution as the one we obtained at first – we have simply chosen to *restate the constant* C , in order to remove an unnecessary explicit numeric constant term from the solution. We can say in our defence that in any situation where we have some useful information, such as a boundary condition, or another fact which allows us to obtain an actual value for the constant of integration, either of these two solutions will allow us to incorporate this information, with just a difference of $1/2$ in the value we actually assign to our constant.

A more detailed discussion of the occasional need to restate the constant, which sometimes arises from using the substitution technique, can be found in the notes at the end of [Appendix 5 Section 7](#).

Example 5: In the Same Vein but with a Different Substitution

In the discussion of [Example 4](#) above, I deliberately chose a substitution which simplified the denominator of the integrand completely. This choice, however, led to the requirement for a restatement of the constant of integration at the end of the process. For those among you who consider this ugly and undignified, here is an alternative solution using a different substitution which avoids this regrettable necessity. It does, however, introduce some extra complication to the process, as you will see.

This time, we will approach the integral using the substitution $u = x^2$. Just as before, this will give us the following:

$$\frac{du}{dx} = 2x, \text{ so that } dx = \frac{1}{2x} du$$

Our integral will thus be transformed as follows:

$$\int \frac{x^3}{2x(u+1)} du$$

Carrying out some cancellation on the terms in x and moving the constant factor of $1/2$ outside the integral sign, we then obtain:

$$\frac{1}{2} \int \frac{u}{u+1} du$$

This cannot be reduced to an easily integrable form with the same simple method we were able to deploy with our earlier substitution. However, there is an algebraic device which can help us out of our bind in cases like this.

We first note that the denominator of the rational expression in u is $u+1$. If we add 1 to the numerator, we will turn our expression into something which can be reduced to 1. However, in order to avoid changing the value of the numerator, we must compensate for the addition of 1 by subtracting 1. You may not think this actually gets us anywhere – but it alters the presentation of our expression in just such a way as to make it possible to change it into something we can handle more easily. The algebra works as follows:

$$\frac{u+1-1}{u+1} = \frac{u+1}{u+1} - \frac{1}{u+1} = 1 - \frac{1}{u+1}$$

(This technique will be used again elsewhere in this book, and I give a symbolic demonstration of its use, which allows it to be applied in many similar situations, in [Appendix 3 Section 2](#).)

Now this lets us restate our integral in u in a form very similar to the one we obtained using our first substitution, namely:

$$\frac{1}{2} \int \left(1 - \frac{1}{u+1} \right) du$$

which integrates, similarly to the way it did before, to give the following solution in terms of u :

$$\frac{1}{2}(u - \ln|u + 1|) + C$$

Here comes the really interesting difference, however: on reversing our substitution of $u = x^2$, we now obtain the following solution in terms of x :

$$\frac{1}{2}(x^2 - \ln(x^2 + 1)) + C$$

This was the expression we arrived at previously after we had gathered all the constant elements of the solution into one single constant of integration.

So we see that by using a substitution of $u = x^2 + 1$, we had to carry out a little further work to remove an unnecessary constant term; using $u = x^2$, however, produced a constant-free expression (apart, of course, from the ever-necessary C). I will now make two digressions in order to explore some general conclusions which we are able to draw from this exploration of a rational integrand. These digressions entail some symbolic reasoning in order to establish the generality of the results they propose.

What Kind of Integral Produces an Explicit Constant in the Result?

(Readers should be aware that the remainder of this section constitutes a digression into realms of abstract reasoning inspired by the results we have just obtained. If you are not in the mood for this, or perhaps don't feel it will be relevant to your needs, then there is nothing to prevent you from proceeding straight on to [Section 8](#) at this point. But don't be too hasty – you might find it interesting and useful after all!)

Is there anything of a general nature we might be able to say about the circumstances in which integration by substitution leads to a solution containing an explicit constant term?

One possibility certainly arises from the first approach we took to [Example 4](#) above: in this case, a substitution of the form $u = f(x) + a$ was chosen; furthermore, the solution to the integral in terms of u contained an actual term of the form ku , which was replaced by $k(f(x) + a)$ or $kf(x) + ka$ on reversing the substitution. This provided a constant term ka in the final solution in terms of x , which led to my discussion about its inclusion in the standard constant of integration – the process I described as “restating the constant”.

We can generalise a little from this observation to suggest the following rule: if we integrate using a substitution of the form $u = f(x) + a$ (where a is a constant) and we find that the solution in terms of u contains a term of the form ku^n , where k is a constant and n is an integer greater than or equal to 0, then the final solution in terms of x will contain a constant term which we will have to subsume in the constant of integration by restating that constant.

Why would you be expected to believe this? Well, if our solution in terms of u contains a term of the form ku^n , then on reversing the substitution, this will turn into $k(f(x) + a)^n$. The Binomial Theorem tells us that the last term in the expansion of the bracketed factor which is raised to the power n will be a^n . Multiplying out using the constant k will produce

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a term ka^n . Since both k and a are assumed to be constants, the composite term will certainly also be a constant. Hence, a constant term will be present in the final solution in terms of x .

Just to tidy matters up, however, we also need to ask the question: why should the power n have to be an integer greater than or equal to 0? First of all, assuming that $n \geq 1$ allows us to quote the Binomial Theorem. Furthermore, if we allow n to be 0, then u^n becomes equal to 1, so that ku^n is equal to k , which is of course a constant.

However, if $n < 0$, then we have:

$$ku^n = \frac{k}{u^m}, \text{ where } m = -n$$

On reversing the substitution, this will become:

$$\frac{k}{(f(x) + a)^m}$$

which contains at least some terms in x in its denominator as well as any constant which arises from the expansion of the bracketed item – so the entire expression may not be reducible in such a way as to produce a constant term.

Can we go any further and deduce some condition which should be fulfilled by the integral in x which we wish to evaluate in order for an explicit constant to be produced? Well, we can say a certain amount; and a certain amount is all I intend to trouble you with here.

We saw above that a substitution of the form $u = f(x) + a$ and a term in the solution in u of the form ku^n , where n is greater than or equal to 0, will produce an explicit constant term in the final solution. Now we can certainly say that one way of satisfying these conditions will be to have a term of the form Ku^{n-1} in the integral in terms of u which we obtain after making the substitution. This is because such a term would integrate to produce the following term in the solution in u :

$$\frac{K}{n}u^n, \text{ which has the form } ku^n \text{ where } k = \frac{K}{n}$$

Let's pause a moment here to note that we can rephrase our conclusion here a little to give us some useful insight. If our integral in u contains a term of the form Ku^N , then it will integrate to give us a term of the form ku^n , where:

$$k = \frac{K}{N}, \text{ and } n = N + 1$$

Since this is so, I will drop the capital letters in my description of the term to be integrated, and simply revert to describing it as a term of the form ku^n .

Given that our substitution is of the form $u = f(x) + a$, we can deduce the following:

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$$\frac{du}{dx} = f'(x), \text{ so that } du = f'(x)dx$$

We can therefore say that if we start with an integral of the form:

$$\int k(f(x) + a)^n f'(x) dx$$

and approach it using the substitution $u = f(x) + a$, then (bearing in mind the expression for du which we obtained above) our transformed integral in terms of u will be of the form:

$$\int ku^n du$$

The reasoning I have set out above predicts that this will yield an explicit constant term as part of its final solution in terms of x .

This is as far as I intend to press this question here. No doubt there will be other ways of obtaining an explicit constant in the solution to an integral, but we have at least been able to describe one possible way this can happen.

In conclusion, let's just pause a moment to notice that the form of integral which we have decided will produce an explicit constant in its solution (with the appropriate substitution – this is a key requirement – is of the standard form for integration by substitution, namely the form:

$$\int KF'(G(x))G'(x)dx$$

where:

$$K = k, F'(x) = x^n, G(x) = f(x) + a, \text{ and } G'(x) = f'(x)$$

Our substitution here should be $u = G(x)$, or, to put it another way, $u = f(x) + a$.

Does Example 4 Illustrate a New Class of Integral Suitable for Substitution?

Having worked through [Example 4](#) above in some detail, we might now be inspired to ask whether it exemplifies a new class of integral which is susceptible to the substitution technique.

In order to answer this speculation, let's take a look at the more general integral:

$$\int \frac{x^m}{x^n + a} dx$$

and try to find out the circumstances in which we can integrate this using substitution.

In line with our first approach to [Example 4](#), let's decide that we will try to use the substitution $u = x^n + a$. From this, we can infer that:

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$$\frac{du}{dx} = nx^{n-1}, \text{ so that } dx = \frac{1}{nx^{n-1}} du$$

Making the substitutions we have just set out in order to obtain an integral with respect to u , we will obtain the following:

$$\int \frac{x^m}{u \cdot nx^{n-1}} du$$

Applying the rules of indices to the x terms in the numerator and denominator, this can be rewritten as follows:

$$\int \frac{x^{m-n+1}}{nu} du$$

Bearing in mind (again in the light of our work on [Example 4](#)) that our substitution formula can be rearranged to give $x^n = u - a$, we can draw the conclusion that the remaining instance of the variable x can easily be taken out of the picture if the numerator of our integrand can be made equal to x^n (in which case we can use the above rearrangement of the substitution formula to replace it with $u - a$).

This equivalence will hold whenever:

$$m - n + 1 = n, \text{ or by rearrangement, } m = 2n - 1$$

When this is the case, our integral becomes:

$$\int \frac{u - a}{nu} du$$

We can easily convert this into a difference of two fractions, which makes it fairly easy to integrate:

$$\int \left(\frac{1}{n} - \frac{a}{nu} \right) du = \int \left(\frac{1}{n} - \frac{a}{n} \cdot \frac{1}{u} \right) du$$

This integrates quite simply to:

$$\frac{u}{n} - \frac{a}{n} \ln|u| + C = \frac{1}{n}(u - a \ln|u|) + C$$

Taking the final step of reversing the substitution, we have the following solution in terms of x :

$$\frac{1}{n}(x^n + a - a \ln|x^n + a|) + C$$

Finally, we can (according to our reasoning above) subsume the constant term in a into the constant of integration C , to arrive at the following slightly simpler solution:

$$\frac{1}{n}(x^n - a \ln|x^n + a|) + C' \text{ where } C' = C + \frac{a}{n}$$

We have therefore established that any integral of the following form:

$$\int \frac{x^{2n-1}}{x^n + a} dx$$

can be integrated using the substitution $u = x^n + a$, and we have even been able to derive the general solution to an integral of this form.

Before leaving this topic, we might just pause to remark that [Example 4](#) above, of course, fits this form exactly, with $n = 2$ and $a = 1$.

(We can actually go a little further in specifying the general form of integral which can be solved by substitution in the way we have been doing here: we can in fact solve integrals of the form:

$$\int \frac{ax^{2n-1}}{bx^n + c} dx$$

using either $u = bx^n + c$ or $u = bx^n$ as our substitutions. In order to avoid a further exhibition of symbol manipulation here, I have chosen instead to give the general solution to integrals of this form in [Appendix 5 Section 4](#).)

9 Power Substitution

In this section, we will examine one particular kind of algebraic substitution which is so useful in dealing with integrals of a certain type that it is sometimes given its own particular name – power substitution. In order to introduce it, I will re-examine an example which we addressed as [Section 3 Example 6](#) above and evaluate it using this new approach. As usual, I will then present a series of additional examples to illustrate the method fully.

We will look again, then, at the following integral:

$$I = \int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$$

In effect, what we did in [Section 3](#) was to use the fact that this integral contains a “function-derivative pair” (subject to the use of a constant factor) in the form of the $x^{1/2}$ within the sine function and the $x^{-1/2}$ which constitutes the denominator. Using the normal rule for differentiating a power of x , we can see that:

$$\frac{d}{dx} x^{1/2} = \frac{1}{2} x^{-1/2}$$

Since I explained in detail how the integrand can be considered to contain this function-derivative pair, as well as the need to justify this introducing a previously invisible constant

factor, in the note on this example in [Section 4](#) above, there's probably no need to labour this point further.

So how can we approach this integral using a different type of substitution-based approach? Bearing in mind the kind of reasoning we saw in the preceding section on algebraic substitutions, we might start by asking: what it is about this integral which contributes most to its difficulty?

One answer might be to point to the square roots of x in the numerator and denominator. So now it would be fair to ask: how might we use a substitution to make these disappear from the problem? One approach might be to substitute for x something squared – let's say, for example, $x = u^2$.

As is frequently the case, we can get to this “reversed” substitution by reasoning that we want to make the square root of x be represented in the transformed integral by u . We therefore say that we want to see:

$$u = \sqrt{x} \text{ so that } u^2 = x$$

The second formulation, of course, means that we will not have to differentiate any square root expressions in obtaining our expression for dx .

If we make this substitution, then, in addition to transforming the integral into a function of u which contains no square roots, we can also conclude the following:

$$\frac{dx}{du} = 2u \text{ so that } dx = 2u du$$

Now this is really not a million miles from the reasoning we used (and which I have tried to set out) when we evaluated this integral the first time. Indeed, the only real difference is that instead of choosing a substitution of $u = \sqrt{x}$, we have simply rearranged this in terms of x to read $x = u^2$. This inversion has also led us to consider the derivative of x as a function of u - that is, the derivative dx/du - in order to find our formula for dx .

Putting both our chosen substitution and our expression for dx back into the original integral, we find it transformed as follows:

$$\int \frac{\sin u}{u} \cdot 2u du = \int 2 \sin u du$$

As we saw the first time round, this integrates very easily to give the solution:

$$-2 \cos u + C = -2 \cos \sqrt{x} + C$$

The final form in terms of x is, of course, obtained by rewriting our substitution formula to obtain an expression for u in terms of x . Our substitution formula is:

$$x = u^2 \text{ so that } u = \sqrt{x}$$

Of course, this is exactly the result we obtained in [Section 3 Example 6](#), where we made a direct u -substitution. All we have done differently here is to express x , the original variable, as a *power* of u , so that the square roots can be removed when we make the substitution and simplify.

Some think that this is a more intuitive and appealing way of dealing with integrals containing roots of x ; the method can also be used to handle integrals which do not contain “function-derivative pairs”, where substitution in the form we have seen it so far might not help us. We have begun with a very simple example, where our power-substitution was more or less equivalent to our original u -substitution in the amount of effort it required. We now need to look at some more examples in order to bring out the real “power” of this substitution technique.

Example 1

This example will feature another integral which contains a square root, but where no obvious “function-derivative pair” can be spotted. Our integral will be as follows:

$$I = \int \frac{1}{1 + \sqrt{x}} dx$$

As in the introduction to this section, we will again aim to apply the substitution $x = u^2$. This will, as above, give us the following two useful consequences:

$$\sqrt{x} = \sqrt{u^2} = u$$

and

$$\frac{dx}{du} = 2u \text{ so that } dx = 2u du$$

These two pieces of information allow us to rewrite our integral as follows:

$$\int \frac{1}{1 + u} \cdot 2u du = 2 \int \frac{u}{1 + u} du$$

Of course, the new integral in terms of u – even after moving the factor of 2 outside the integral sign – doesn’t look much easier to deal with. However, we can find a way to make it a little simpler by performing the neat algebraic “trick” of adding and subtracting 1 to the numerator. This does not change its value, since it is always true to say that $a + 1 - 1 = a$!

Doing this alters the integrand in the following way:

$$\frac{1 + u - 1}{1 + u} = \frac{1 + u}{1 + u} - \frac{1}{1 + u} = 1 - \frac{1}{1 + u}$$

It could be argued that the identity:

Integration by Substitution

$$\frac{x}{x+1} = 1 - \frac{1}{x+1}$$

should be considered as a standard fact of algebra – and you may already have learnt it as such. If not, I hope the argument I have given here about the possibility of adding and subtracting 1 to the numerator will be convincing enough to allow us to use it to reduce the integrand to an easily handled form. In any case, I set out a slightly more general argument about this method of converting rational functions to more agreeable forms in [Appendix 3 Section 2](#).

This immediately turns the integrand into a simple difference of two terms, each of which we can integrate quite easily. The result is:

$$\int \left(1 - \frac{1}{1+u}\right) du = u - \ln|1+u| + C$$

Restoring the factor of 2 which we moved outside the integral sign a moment ago, and reversing the substitution using the expression for u in terms of x which we noted at the outset, we have a final solution as follows:

$$I = 2\sqrt{x} - 2\ln|1 + \sqrt{x}| + C$$

Example 2

We will now move on to look at an integral which contains a more complicated square root, to see how we can still apply power substitution to evaluate it. Our example will be as follows:

$$I = \int \frac{3x+2}{\sqrt{x-9}} dx$$

We will aim to substitute the entire expression under the square root sign with u^2 , in the hope of making the square root sign disappear completely in the transformed integral. Our proposed substitution will therefore be represented by the expression:

$$x - 9 = u^2$$

We could rewrite this as an expression for x in terms of u using simple algebraic rearrangement to give us:

$$x = u^2 + 9$$

We should furthermore confirm that this does indeed deal with the undesirable square root sign by noting that:

$$\sqrt{x-9} = \sqrt{u^2} = u$$

Integration by Substitution

This is exactly the same approach we took to [Example 1](#) – choose our substitution so that we end up with u^2 under the square root sign, which allows us to reduce the entire square root expression to u alone.

One more step is called for – obtaining an expression for dx . This can be obtained by differentiating our expression for x in terms of u as follows:

$$\frac{dx}{du} = 2u \text{ so that } dx = 2u du$$

We are now in a position to put all these facts into our original integral in order to end up with an integral in terms of u . Using each one of the facts we noted above, the transformed integral will be as follows:

$$\int \frac{3(u^2 + 9) + 2}{u} \cdot 2u du = \int \frac{6u^3 + 58u}{u} du = \int (6u^2 + 58) du$$

The final right-hand side integral can easily be evaluated term by term to give the solution:

$$2u^3 + 58u + C$$

We can now reverse the substitution to get an expression in terms of x , which will look like this:

$$2(\sqrt{x-9})^3 + 58\sqrt{x-9} + C$$

We may feel that it's good enough to leave the solution in this form. However, we can if preferred rewrite the whole expression using fractional indices, with the following effect:

$$2((x-9)^{1/2})^3 + 58(x-9)^{1/2} + C$$

The first term will simplify using the laws of indices, and we will be left with the following final solution:

$$I = 2(x-9)^{3/2} + 58(x-9)^{1/2} + C$$

Example 3

This time, we will consider an example which has some differences from both the previous two, but which will still respond well to power substitution. Our integral will be:

$$I = \int \frac{1}{x^{2/3} - x^{1/3}} dx$$

How can we apply power substitution to this integral? A clue might be provided if we rewrite it without the use of fractional indices, as follows:

Integration by Substitution

$$I = \int \frac{1}{(\sqrt[3]{x})^2 - \sqrt[3]{x}} dx$$

We can now see that the denominator of the integrand contains two cube roots of x : our first instinct, therefore, might be to attempt to make these go away by a suitable power substitution. If we choose $x = u^3$, by analogy with our earlier approach to square roots in our integrand, then we will obtain the following equivalences for the two indexed terms in x in our denominator:

$$x^{1/3} = (u^3)^{1/3} = u$$

$$x^{2/3} = (u^3)^{2/3} = u^2$$

These follow directly from the laws of indices. We also have the following expression for u in terms of x , which will come in handy when we reverse our substitution later:

$$u = \sqrt[3]{x} = x^{1/3}$$

As for our expression for dx , we have the following:

$$\frac{dx}{du} = 3u^2 \text{ so that } dx = 3u^2 du$$

Substituting all these facts back into our original integrand, we will obtain this in terms of u :

$$\int \frac{1}{u^2 - u} \cdot 3u^2 du = 3 \int \frac{u^2}{u(u-1)} du = 3 \int \frac{u}{u-1} du$$

This is a little neater – in particular, all the indexed terms have now disappeared from view. However, we still can't integrate it directly as it stands.

A similar approach to converting the rational integrand in u to that which helped us out in [Example 1](#) above might get us out of trouble: we will change the appearance (leaving the value unchanged) of the numerator by adding and subtracting 1. This will give us:

$$\frac{u}{u-1} = \frac{u-1+1}{u-1} = \frac{u-1}{u-1} + \frac{1}{u-1} = 1 + \frac{1}{u-1}$$

Just as in [Example 1](#), the integration is now an easy term-by-term task, as follows:

$$\int \left(1 + \frac{1}{u-1}\right) du = u + \ln|u-1| + C$$

Putting back the factor of 3 which we took outside the integral sign and reversing the substitution, we arrive at the following final solution:

$$I = 3x^{1/3} - 3\ln|x^{1/3} - 1| + C$$

Example 4

For the concluding example in this section, we will take a case where it is not so easy to see what substitution will help us to perform the integration. This somewhat trickier example will be the following integral:

$$I = \int \frac{\sqrt[3]{x} + \sqrt{x}}{\sqrt[3]{x}} dx$$

We will no doubt immediately become concerned by the fact that the integrand contains *both* cube roots and a square root. We therefore can't make the same quick choice of power substitution we were able to make in previous examples. So how should we proceed?

The first thing we might do is to ask ourselves this question (recalling as we do the fractional index representations of the square and cube roots) - is there a whole number value of n for which both:

$$(u^n)^{1/2} \text{ and } (u^n)^{1/3}$$

are equivalent under the laws of indices to something of the form u^k , where k is also a whole number? If we can find such a number, then a power substitution of $x = u^n$ ought to remove from view both kinds of root in our original integral.

Bearing in mind, as ever in this context, the laws of indices, we can say that this is equivalent to asking – is there a whole number value of n for which both:

$$\frac{n}{2} \text{ and } \frac{n}{3}$$

take whole number values?

To simplify our question once more, this is tantamount to asking for a number n which is exactly divisible by both 2 and 3. The answer is quite simple – such a number is provided by the lowest common multiple of 2 and 3, which is, of course, 6. We therefore propose the following substitution:

$$x = u^6$$

This allows us to substitute for our roots of x as follows:

$$\sqrt{x} = x^{1/2} = (u^6)^{1/2} = u^3$$

$$\sqrt[3]{x} = x^{1/3} = (u^6)^{1/3} = u^2$$

As usual, an expression for u in terms of x will come in useful when we are ready to reverse the substitution. This expression is:

$$u = x^{1/6}$$

Integration by Substitution

Additionally, we note that:

$$\frac{dx}{du} = 6u^5 \text{ so that } dx = 6u^5 du$$

We can now put all this back into our integral, and find it transformed as follows:

$$\int \frac{u^2 + u^3}{u^2} \cdot 6u^5 du = \int (1 + u) \cdot 6u^5 du = 6 \int (u^5 + u^6) du$$

The final right-hand side integral is easily integrated term by term to produce:

$$\frac{1}{6}u^6 + \frac{1}{7}u^7 + C$$

and applying the factor of 6 which we took outside the integral sign gives us the slightly neater:

$$u^6 + \frac{6}{7}u^7 + C$$

We are now ready to reverse the substitution, to obtain the following final solution:

$$I = x + \frac{6}{7}x^{7/6} + C$$

(Note that $(x^{1/6})^6 = x$, by the laws of indices.)

This is really a surprisingly neat result obtained without too much difficulty, from what seemed at first to be a seriously intractable integral.

The alert reader may already have noticed that the original integrand could have been simplified, in order to produce something which would be immediately integrable – I drew a discreet veil over this possibility in order to use the example to illustrate a particular point about power substitution. However, in case the solution just obtained should seem to strain credulity through its simplicity in comparison with the integrand we began with, I will confirm the result by performing the simplification I have mentioned, and checking that with its assistance, we will get the same result as power substitution gave us.

In order to simplify the integrand, we will rewrite it as follows:

$$\frac{\sqrt[3]{x} + \sqrt{x}}{\sqrt[3]{x}} = \frac{\sqrt[3]{x}}{\sqrt[3]{x}} + \frac{\sqrt{x}}{\sqrt[3]{x}} = 1 + \frac{x^{1/2}}{x^{1/3}}$$

The final term here can be simplified using the laws of indices – it will reduce to x raised to the power of the difference between $1/2$ and $1/3$, which is as follows:

$$\frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \frac{1}{6}$$

Integration by Substitution

Our integral has now simplified to:

$$I = \int (1 + x^{1/6}) dx$$

Which we can simplify term-by-term (the second term will respond to the normal rule for integrating x^n) to give us the following result:

$$I = x + \frac{6}{7}x^{7/6} + C$$

which is exactly the same as the one we obtained by the more roundabout power-substitution approach.

An Explanation of the Principle

In each of the examples presented here, we have used an algebraic substitution in order to eliminate difficult roots of x from various integrands. The choice of substitution was made in each case so as to ensure that the transformed integral in terms of u had all its roots of x transformed into simpler expressions in terms of u .

In [Example 1](#), where we had to deal with an integrand containing $\sqrt{x}$, we chose to make the substitution $x = u^2$, so that in transforming the integral to one in terms of u , we found that:

$$\sqrt{x} = x^{1/2} = (u^2)^{1/2} = u^1 = u$$

Similar reasoning in [Example 3](#) led us to deal with the cube roots of x in the integrand by substituting $x = u^3$.

[Example 2](#) presented us with a linear expression in x , namely $x - 9$, under a square root sign. Here we chose the substitution $x - 9 = u^2$, so that the entire square root expression could be transformed into a simple instance of u . This last example indicates one reason why we choose to express power substitutions in the form $f(x) = g(u)$, rather than just setting out a simple expression for u : we need to ensure that the entire root of some function of x is eliminated by being transformed into a simple expression in terms of u (frequently u on its own!). It may be easier to see this process at work by describing the substitution the way we have been doing in these examples. There is another reason, which is perhaps more convincing, and I will offer this in a moment.

Of course, there is always a “payback” when choosing what seems like a really useful substitution – and that comes in the form of the expression for dx which we must obtain before being able to create the transformed integral. If we phrase the substitution as, for example, $x = f(u)$, we can then note the following general consequences:

$$\frac{dx}{du} = f'(u) \text{ so that } dx = f'(u) du$$

By way of a concrete case, let's imagine we have to integrate a function which contains the term $x^{1/5}$. (We could obviously integrate this directly using the standard rule for integrating

powers of x , but let's see what would happen if we were to use power substitution instead.) Using the approach I have just outlined, we would choose a power substitution of $x = u^5$, from which we can obtain the following:

$$\frac{dx}{du} = 5u^4 \text{ so that } dx = 5u^4 du$$

The integral of our term in x would therefore be transformed as follows:

$$\int x^{1/5} dx \text{ becomes } \int u \cdot 5u^4 du = \int 5u^5 du$$

In general, when we choose a substitution of the form $x = u^n$, we obtain $dx = nu^{n-1} du$. Since we have seen that power substitution proves very useful when we have to integrate rational functions of roots of x - which are not amenable to term-by-term integration - it is very reassuring to find that our expression for dx just contributes an additional power of u to the numerator of a rational expression, rather than contributing something awkward and undesirable to the denominator.

For example, if we were to express the substitution in what we might feel to be the more natural way - in this case as $u = x^{1/5}$, we would then have to write down the following in order to obtain our expression for dx :

$$\frac{du}{dx} = \frac{1}{5} x^{-4/5} = \frac{1}{5x^{4/5}} \text{ so that } dx = 5x^{4/5} du = 5u^4 du$$

This does get us to the same place as our earlier reasoning, but getting there is really not at all as simple or elegant.

Two more points arising from this set of examples are worth bringing out. First, in [Example 2](#), our integrand contained the expression $x - 9$ under the square root sign. We chose a substitution of $x - 9 = u^2$. This did not make things any more complicated than a simple power substitution of the form $x = u^n$. This is because, after rewriting the substitution formula as $x = u^2 + 9$, we were still able to obtain the simple relation $dx = 2u du$, since the constant term of 9 did not make any contribution to the expression for dx/du .

Finally, I would point out once more the special reasoning we applied in choosing the power substitution for [Example 4](#), where we had both cube roots and a square root of x in the integrand. Expressing these roots in the form $x^{1/3}$ and $x^{1/2}$, we found that a substitution of the form $x = u^n$ could be chosen, where n was selected to be the lowest common multiple of 2 and 3 - namely, 6. This choice, after substitution, resolved all the roots of x in the integrand into whole number powers of u , allowing us to complete the job with just the help of a little algebra.

I evaluate a couple of integrals containing nested roots of x in the following section. Additionally, [Part 3](#) of this book contains some more examples of power substitution in conjunction with specific algebraic techniques such as partial fractions and a form of polynomial division.

10 Integrating Nested Roots with Power Substitution

In this section, I will present three examples where functions containing nested roots are integrated successfully using power substitution. I have chosen to include these in a separate section since such integrals might be regarded as relatively rare. However, it is important to include some treatment of these integrals – first of all for the sake of complete coverage, but also in order to demonstrate the wide range of applicability of the power substitution technique.

Example 1

Here we will consider the following integral:

$$I = \int \sqrt{5 + \sqrt{x}} \, dx$$

We will approach this using power substitution, starting off by simply trying to do away with the outer square root by choosing an appropriate substitution. Substituting as follows:

$$u^2 = 5 + \sqrt{x}$$

would convert the outer square root expression into $\sqrt{u^2}$ - or in other words, simply u .

That seems a very appropriate starting point, but what does it imply about the expression for dx which we will now need? Bearing in mind that we have generally expressed power substitutions in the form $x = f(u)$ in order to differentiate and obtain the desired expression, let's rearrange our chosen substitution into this form. We first obtain:

$$\sqrt{x} = u^2 - 5$$

by subtracting 5 from each side of the substitution expression. Squaring both sides of this result will bring us to the following result:

$$x = (u^2 - 5)^2$$

We are now in a position to obtain an expression for dx/du . Using the Chain Rule on the above expression in terms of u , we obtain:

$$\frac{dx}{du} = 2(u^2 - 5) \cdot 2u = 4u(u^2 - 5) = 4u^3 - 20u$$

(We could alternatively multiply out the squared bracketed term in our expression for x to obtain:

$$x = u^4 - 10u^2 + 25$$

which differentiates term by term to give the same result as we obtained using the Chain Rule.)

Integration by Substitution

From either method, we can conclude that our expression for dx is as follows:

$$dx = (4u^3 - 20u)du$$

Putting all this back into our original integral, we obtain the following relatively undaunting integral in terms of u :

$$\int u(4u^3 - 20u)du = \int (4u^4 - 20u^2)du$$

Despite the fact that we seem to have accumulated at least one fairly high power of u in a rather short time, this will integrate very easily term by term to give us this solution in terms of u :

$$\frac{4}{5}u^5 - \frac{20}{3}u^3 + C$$

We are now ready to reverse the substitution. We first note that, from our original substitution formula:

$$u^2 = 5 + \sqrt{x}$$

we are able to deduce the following expression for u :

$$u = \sqrt{5 + \sqrt{x}}$$

Our solution in terms of u can therefore be rewritten as:

$$\frac{4}{5}\left(\sqrt{5 + \sqrt{x}}\right)^5 - \frac{20}{3}\left(\sqrt{5 + \sqrt{x}}\right)^3 + C$$

A final refinement might be to rewrite our original nested square root integrand using fractional power notation, as follows:

$$\sqrt{5 + \sqrt{x}} = (5 + \sqrt{x})^{1/2}$$

This allows us to use the laws of indices to provide a final solution which contains fewer square root signs. This final solution is as follows:

$$I = \frac{4}{5}(5 + \sqrt{x})^{5/2} - \frac{20}{3}(5 + \sqrt{x})^{3/2} + C$$

Example 2

For this second example, we will look at an integral of slightly increased complexity:

Integration by Substitution

$$I = \int \sqrt[3]{1 + \sqrt{x-3}} dx$$

Here, the outer root is a cube root, and the inner root contains not merely x , but the linear expression $x - 3$. However, it will prove to be susceptible to a similar approach to that of [Example 1](#).

We will start by choosing a substitution which will allow us to remove the outer cube root. To this end, and in direct analogy to the choice we made in [Example 1](#), we will substitute as follows:

$$u^3 = 1 + \sqrt{x-3}$$

This will, of course, transform the integrand into $\sqrt[3]{u^3}$, or simply u . To obtain our expression for dx , we will rearrange the substitution expression as before to make it into an expression for x in terms of u . We start by subtracting 1 from both sides to obtain:

$$\sqrt{x-3} = u^3 - 1$$

Squaring both sides now leads to the following:

$$x - 3 = (u^3 - 1)^2$$

and we can now add 3 to both sides to end up with:

$$x = (u^3 - 1)^2 + 3$$

Differentiating using the Chain Rule allows us to conclude that:

$$\frac{dx}{du} = 2(u^3 - 1) \cdot 3u^2 = 6u^2(u^3 - 1) = 6u^5 - 6u^2$$

(Once again, we can briefly note the alternative approach of multiplying out the squared bracketed term and adding the final term of 3 in the expression for x to obtain:

$$x = u^6 - 2u^3 + 4$$

This, of course, differentiates term by term to give the same result as the Chain Rule.)

We can now write down our expression for dx as follows:

$$dx = (6u^5 - 6u^2)du$$

Plugging all of this back into the original integral gives us the following integral in terms of u :

$$\int u(6u^5 - 6u^2)du = \int (6u^6 - 6u^3)du$$

Integration by Substitution

This can be integrated term by term to give the following result:

$$\frac{6}{7}u^7 - \frac{6}{4}u^4 + C = \frac{6}{7}u^7 - \frac{3}{2}u^4 + C$$

As before, we are almost finished. In order to show the effect of reversing the substitution, we first note that our substitution expression:

$$u^3 = 1 + \sqrt{x-3} \text{ gives us } u = \sqrt[3]{1 + \sqrt{x-3}}$$

Reversing the substitution thus gives the following solution in terms of x :

$$\frac{6}{7} \left(\sqrt[3]{1 + \sqrt{x-3}} \right)^7 - \frac{3}{2} \left(\sqrt[3]{1 + \sqrt{x-3}} \right)^4 + C$$

We can rewrite this as in [Example 1](#) using the laws of indices to obtain the following final solution:

$$I = \frac{6}{7} (1 + \sqrt{x-3})^{7/3} - \frac{3}{2} (1 + \sqrt{x-3})^{4/3} + C$$

Example 3

In this final example, we will increase the level of complexity still further, by considering the integration of a function comprising *three* levels of nesting. Our integral will be:

$$I = \int \sqrt{2 + \sqrt{4 + \sqrt{x}}} dx$$

In line with our approach to the two preceding examples, we will start with a substitution which will allow us to remove the outermost root from consideration. This substitution will be:

$$u^2 = 2 + \sqrt{4 + \sqrt{x}}$$

As was the case in [Example 1](#), this will transform the integrand into $\sqrt{u^2}$, or simply u .

In order to obtain our expression for dx , we now have to rearrange the substitution formula to make x the subject. This will require more manipulation than in the previous examples. To begin with, we can subtract 2 from both sides in order to obtain:

$$\sqrt{4 + \sqrt{x}} = u^2 - 2$$

Squaring both sides will take us a stage further, and give us this equation:

Integration by Substitution

$$4 + \sqrt{x} = (u^2 - 2)^2$$

We should now be able to see that the end is in sight. Subtracting 4 from both sides gives us:

$$\sqrt{x} = (u^2 - 2)^2 - 4$$

and a final squaring of both sides brings this expression for x in terms of u :

$$x = ((u^2 - 2)^2 - 4)^2$$

In order to differentiate this to find dx/du , we need to use the “three-level” version of the Chain Rule, which states that if:

$$y = h(g(f(x))) \text{ then } \frac{dy}{dx} = h'(g(f(x))) g'(f(x)) f'(x)$$

This rule gives us the following derivative:

$$\frac{dx}{du} = 2((u^2 - 2)^2 - 4) \cdot 2(u^2 - 2) \cdot 2u = 8u(u^2 - 2)(u^4 - 4u^2)$$

(The final factor comes from multiplying out $(u^2 - 2)^2$ and subtracting 4 – in other words by expanding the first bracketed factor which follows the 2 in the first version of the derivative.)

Multiplying out the final right-hand side above gives:

$$8u(u^6 - 4u^4 - 2u^4 + 8u^2) = 8u(u^6 - 6u^4 + 8u^2)$$

and a final step of multiplying out brings us to a useful final expression for the derivative:

$$\frac{dx}{du} = 8u^7 - 48u^5 + 64u^3 \text{ so that } dx = (8u^7 - 48u^5 + 64u^3)du$$

This is the most complicated expression for dx which we have obtained in this section so far – and it contains even higher powers of u than in earlier cases.

(With my customary desire to provide an alternative derivation, particularly for anyone uncomfortable with the version of the Chain Rule I have deployed in this case, let's see what happens when we multiply out the various parts of the expression we obtained for x and then differentiate the result.

We recall first that:

$$x = ((u^2 - 2)^2 - 4)^2$$

so that expanding out the outer squared bracketed item, we obtain for the right-hand side:

$$(u^2 - 2)^4 - 8(u^2 - 2)^2 + 16$$

Integration by Substitution

We can now proceed to expand out the powers of $u^2 - 2$ - using the Binomial Theorem for the fourth power, and the ordinary quadratic expansion in the second case. As a reminder, the appropriate Binomial Theorem result for this situation is as follows:

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

These two expansions give us:

$$(u^2 - 2)^4 = u^8 - 8u^6 + 24u^4 - 32u^2 + 16$$

and

$$(u^2 - 2)^2 = u^4 - 4u^2 + 4$$

Putting these back into the first expansion of our expression for x gives:

$$u^8 - 8u^6 + 24u^4 - 32u^2 + 16 - 8u^4 + 32u^2 - 32 + 16$$

Gathering like terms together, we obtain the following revised expression for x :

$$x = u^8 - 8u^6 + 16u^4$$

From this rather simpler expression for x , we obtain by differentiation:

$$\frac{dx}{du} = 8u^7 - 48u^5 + 64u^3$$

which is exactly the same as the result we derived using the three-level Chain Rule.)

Putting the results of all this reasoning back into our original integral, we will obtain the following integral in terms of u :

$$\int u \cdot (8u^7 - 48u^5 + 64u^3) du = \int (8u^8 - 48u^6 + 64u^4) du$$

This now integrates term by term to give us the following solution in terms of u :

$$\frac{8}{9}u^9 - \frac{48}{7}u^7 + \frac{64}{5}u^5 + C$$

We are now ready to reverse the substitution. We note, in order to do this, that by choosing our substitution as:

$$u^2 = 2 + \sqrt{4 + \sqrt{x}} \text{ we can write } u = \sqrt{2 + \sqrt{4 + \sqrt{x}}}$$

This gives us the following solution in terms of x :

$$\frac{8}{9} \left(\sqrt{2 + \sqrt{4 + \sqrt{x}}} \right)^9 - \frac{48}{7} \left(\sqrt{2 + \sqrt{4 + \sqrt{x}}} \right)^7 + \frac{64}{5} \left(\sqrt{2 + \sqrt{4 + \sqrt{x}}} \right)^5 + C$$

To conclude, we can rewrite this as we have done before using the laws of indices to give this final solution:

$$I = \frac{8}{9} \left(2 + \sqrt{4 + \sqrt{x}} \right)^{9/2} - \frac{48}{7} \left(2 + \sqrt{4 + \sqrt{x}} \right)^{7/2} + \frac{64}{5} \left(2 + \sqrt{4 + \sqrt{x}} \right)^{5/2} + C$$

Some Observations

All the above examples have a good deal in common. In all cases, the most complicated part of the process was the work we had to do to obtain our expression for dx : this always turned out to be a series of powers of u accompanied by their various coefficients. We were always able (frequently by squaring both sides of our rearranged substitution formula) to remove roots of x , and the end result always included some unexpectedly high powers of u .

The substitution expression was chosen with a view to eliminating the outermost root as simply as possible. In general, we were always dealing with integrals of the form:

$$\int (f(x))^{1/n} dx$$

And we chose a substitution of the form $u^n = f(x)$, noting that $(u^n)^{1/n} = u$.

Why did we choose to express the substitution in this way, rather than simply stating it in terms of u alone? In order to see why this was a more fruitful approach, let's take a look at [Example 1](#) with the substitution expressed in this way.

Here we chose the substitution:

$$u^2 = 5 + \sqrt{x}$$

Noting towards the end of the working that this means that:

$$u = \sqrt{5 + \sqrt{x}}$$

so why not just start with this expression instead of the expression for u^2 ?

When we have a substitution expression of this kind, a natural instinct might be to obtain our necessary expression for dx by first evaluating du/dx . In this case, we will first rewrite the expression for u using fractional indices, as follows:

$$u = (5 + x^{1/2})^{1/2}$$

This differentiates using the Chain Rule to give us:

$$\frac{du}{dx} = \frac{1}{2}(5 + x^{1/2})^{-1/2} \cdot \frac{1}{2}x^{-1/2}$$

This already looks unpromising – however if we rewrite it without negative powers, as follows:

$$\frac{du}{dx} = \frac{1}{4\sqrt{x}(5 + \sqrt{x})}$$

and try to express it in terms of u , as follows:

$$\frac{du}{dx} = \frac{1}{4u^2(u^2 - 5)}$$

we do still finally arrive at our original expression for dx , namely:

$$dx = 4u^2(u^2 - 5)du = (4u^3 - 20u)du$$

It could be argued that this was rather more laborious – requiring us to differentiate fractional powers of x and then deal with a disagreeable-looking rational expression before we finally reach our goal – than the approach we took when presenting the example. Indeed, it could be suggested as a general rule in mathematics that one should always adopt the simpler and less error-prone method when a choice between two or more exists; and certainly there is nothing like fractional indices to create potential upsets when differentiating!

I therefore hope that this brief excursion will serve to convince you that power substitution is a convenient and straightforward way to deal with integrands containing nested roots of x .

11 Trigonometric Substitutions

Apart from their interesting symmetry under differentiation, the trigonometric functions also satisfy a whole range of useful identities, such as the well-known equation:

$$\sin^2 x + \cos^2 x = 1$$

These identities, and, it should be noted, simple rearrangements of them, can be used in conjunction with carefully chosen substitutions, to make some really gruesome integrals quite easy to solve. This is a similar approach to that of the substitutions we described above as “algebraic”, with the added advantage that we have a wide range of standard relations between trigonometric functions to help us if we can transform our integrand to take advantage of them.

We’ll illustrate this approach as before with several examples, and accompany each with some remarks on generalisations or other points of interest which may arise.

Example 1

Let’s consider the following integral:

$$I = \int \frac{1}{\sqrt{9 - x^2}} dx$$

Adopting the “reversed substitution” approach of [Example 3 in the previous section](#), we will use the substitution $x = 3 \sin u$. (Remember at this point my earlier remarks about the trial and error which must have led me to making this choice!)

Using this substitution, we will find that:

$$9 - x^2 = 9 - 9\sin^2 u = 9(1 - \sin^2 u) = 9\cos^2 u$$

The final stage in this chain of equalities is obtained by a simple rearrangement of the trigonometric identity we quoted at the start of this section in order to make $\cos^2 x$ the subject. And perhaps now we can see the point of the rather arbitrary decision I made a few lines above to use the particular substitution I chose – it lets us make use of a trigonometric identity to simplify our integrand. This kind of substitution is referred to by some authors as “sine-substitution”, and its success derives from its ability to call on the power of the trigonometric identity I have quoted.

However, the heavy lifting is not yet complete: apart from noting quickly that with the aid of our chosen substitution:

$$\sqrt{9 - x^2} = \sqrt{9\cos^2 u} = 3 \cos u$$

we must now move on to the question of an expression for the “differential” dx . Using our by now familiar procedure, we have:

$$\frac{dx}{du} = 3 \cos u$$

so that $dx = 3 \cos u \, du$.

Plugging these conclusions back into our original integral, we find it transformed into:

$$\int \frac{3 \cos u}{3 \cos u} du = \int 1 du = u + C$$

which is a surprisingly neat and simple result considering the integrand we started out with.

Our final job is to reverse the substitution, which requires us to obtain an expression for u in terms of x . Our original substitution formula can be rearranged quite simply as follows:

$$x = 3 \sin u \text{ so that } \sin u = \frac{1}{3}x \text{ and so } u = \sin^{-1} \frac{1}{3}x$$

This gives us a final result of:

Integration by Substitution

$$I = \sin^{-1} \frac{1}{3}x + C$$

Note

We can generalise this example to the problem of integrating:

$$I = \int \frac{1}{\sqrt{A - x^2}} dx$$

Using the same reasoning as in our concrete example, we will choose the substitution $x = \sqrt{A} \sin u$, which (again, by exactly similar reasoning as in our concrete case) will give us a final solution of:

$$\sin^{-1} \frac{1}{\sqrt{A}}x + C$$

(It's worth pointing out that integrals of this type are often represented in textbooks as having the form:

$$I = \int \frac{1}{\sqrt{a^2 - x^2}} dx$$

I've chosen the form I have in order to emphasise that if A in my notation is not a perfect square, we need not be afraid of using a square root as the coefficient of $\sin u$ in our substitution. I have reverted to the more common form using a^2 in my notes below and in the Appendices.)

As an added bonus, we can also see that if we didn't already know the derivative of the inverse sine function, the following special case of the above general rule:

$$\int \frac{1}{\sqrt{1 - x^2}} dx = \sin^{-1}x + C$$

will give us the information we lacked with the aid of the Fundamental Theorem of Calculus.

Using a Different Trigonometric Identity

The next two examples, which will indicate ways of integrating two further classes of integrand, will feature the use of an additional trigonometric identity:

$$\sec^2 x - \tan^2 x = 1$$

For those unfamiliar with the function $\sec x$, here is its definition:

$$\sec x = \frac{1}{\cos x}$$

Integration by Substitution

The trigonometric identity above hints at some partial symmetry between $\sec x$ and $\tan x$ (which will become more apparent when we mention these functions' derivatives) and we will use rearrangements of it to simplify integrands in a similar way to the method of sine-substitution we used in [Example 1](#). In order to provide as complete an explanation as possible, here are two rearrangements of the identity, each making a different function the subject:

$$\sec^2 x = 1 + \tan^2 x$$

$$\tan^2 x = \sec^2 x - 1$$

(It may be of interest to note that this new identity relating $\tan x$ and $\sec x$ can be derived fairly easily from the more common Pythagorean identity which connects $\sin^2 x$ and $\cos^2 x$ by dividing both sides of this latter identity throughout by $\cos^2 x$.)

In order to use this new identity in the context of integration by substitution, we will need to know what the derivatives of $\sec x$ and $\tan x$ are, to help us create expressions for “differentials”. Since this book are concerned with integration techniques, I will simply quote the derivatives below, and give a sketchy description of how they are obtained.

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

The first derivative is obtained by using the Chain Rule on the definition of $\sec x$: note that as is common with the Chain Rule, our derivative is a product of two functions.

The second one can be confirmed by using the Quotient Rule on the common identity:

$$\tan x = \frac{\sin x}{\cos x}$$

which gives us a squared term in the denominator of a fraction. The result $\sec^2 x$ arises from the definition of $\sec x$ which we gave earlier.

A potentially very useful alternative form of the derivative of $\tan x$ arises from the new trigonometric identity we introduced at the start of this section, in the rearrangement which makes $\sec^2 x$ the subject. This provides the following possible form for the derivative:

$$\frac{d}{dx} \tan x = \sec^2 x = \tan^2 x + 1$$

I have altered the order of the two terms in the final expression in order to stress the curious fact that under differentiation, the function $\tan x$ actually produces a *higher* power of itself. This contrasts teasingly with the power-reduction behaviour of functions of the form x^n under differentiation.

Integration by Substitution

Since we will need it later, we will also quote the integral of $\sec x$, and use it as a standard result:

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

This integral can actually be evaluated using substitution, but only with the prior use of a non-intuitive piece of algebraic trickery. For those who would like to see this piece of mathematical swashbuckling, I set it out in [Appendix 5 Section 6](#).

Example 2

Here, we will consider the integral:

$$I = \int \frac{1}{\sqrt{5 + x^2}} \, dx$$

This time, in order to be able to use our new trigonometric identity, we will try the substitution $x = \sqrt{5} \tan u$ (being resolutely unafraid of using a square root, since 5 is not a perfect square)

Using this substitution, we will find that:

$$5 + x^2 = 5 + 5\tan^2 u = 5(1 + \tan^2 u) = 5\sec^2 u$$

by simple rearrangement of our new trigonometric identity to make $\sec^2 u$ the subject. As one might imagine, this approach is sometimes referred to as “tangent-substitution”.

We now note the effect of our substitution on the denominator of our integrand:

$$\sqrt{5 + x^2} = \sqrt{5\sec^2 u} = \sqrt{5} \sec u$$

Moving on towards an expression for dx (and here, we can use the derivative we quoted above):

$$\frac{dx}{du} = \sqrt{5} \sec^2 u$$

so that $dx = \sqrt{5} \sec^2 u \, du$.

Substituting all this back into our original integral, we are left with:

$$\int \frac{\sqrt{5} \sec^2 u}{\sqrt{5} \sec u} \, du = \int \sec u \, du = \ln|\sec u + \tan u| + C$$

Finally, we can reverse the substitution, which requires us to obtain an expression for u in terms of x . We are left with the following result:

Integration by Substitution

$$\ln \left| \sec \tan^{-1} \frac{1}{\sqrt{5}} x + \frac{1}{\sqrt{5}} x \right| + C$$

obtaining the second, simplified, term inside the modulus brackets by keeping in mind that $\tan \tan^{-1} a = a$.

The compound function $\sec \tan^{-1} u$ which appears in our solution above is a rather ugly customer. Fortunately, we can simplify it to obtain a solution much more in keeping with the appearance of our original integrand. (Indeed, there are a number of useful identities which let us simplify compounds of trigonometric functions and inverse trigonometric functions like this one: these are often not well known, but they can be derived without too much difficulty if you know how to go about it. I have quoted some of these identities and given derivations of them in [Appendix 4 Section 3](#).)

I will therefore simply quote here the identity which will let us simplify $\sec \tan^{-1} x$:

$$\sec \tan^{-1} x = \sqrt{x^2 + 1}$$

So, to round off our discussion of the current example of integration, how does this affect our solution? Well, we can now rewrite the first term inside the modulus brackets as follows:

$$\sec \tan^{-1} \frac{1}{\sqrt{5}} x = \sqrt{\frac{1}{5} x^2 + 1} = \sqrt{\frac{x^2 + 5}{5}} = \frac{1}{\sqrt{5}} \sqrt{5 + x^2}$$

(In the final right-hand side, I have reversed the order of x^2 and 5 to match the order in our original integrand.)

We can now rewrite the expression inside the modulus brackets in our first solution as follows:

$$\sec \tan^{-1} \frac{1}{\sqrt{5}} x + \frac{1}{\sqrt{5}} x = \frac{1}{\sqrt{5}} \sqrt{5 + x^2} + \frac{1}{\sqrt{5}} x = \frac{1}{\sqrt{5}} (\sqrt{5 + x^2} + x)$$

This gives us a rewritten solution of:

$$I = \ln \left| \frac{1}{\sqrt{5}} (\sqrt{5 + x^2} + x) \right| + C$$

Now, we have still not quite finished. We must at this point recall the rule of logarithms which states that:

$$\ln(ab) = \ln a + \ln b$$

By this rule, we can rewrite the solution we obtained above as follows:

$$I = \ln \left(\frac{1}{\sqrt{5}} \right) + \ln \left| \sqrt{5 + x^2} + x \right| + C$$

Integration by Substitution

We noted in the preceding section that solutions to integrals which contain explicit constant terms can be rewritten by absorbing the constant terms into the constant of integration. In this case, we can restate the constant as follows:

$$C' = C + \ln\left(\frac{1}{\sqrt{5}}\right)$$

With this restatement of the constant, we have the following final solution to our integral:

$$I = \ln|\sqrt{5+x^2} + x| + C'$$

Note

This last result, and its reduction to a form free of trigonometric functions, is really quite a horror, and this complexity stems from two aspects of our method: first, the derivatives of $\sec x$ and $\tan x$ are not as neatly symmetrical as are those of $\sin x$ and $\cos x$, which means that we were unable to cancel elements of the numerator and denominator of our substituted integral quite as comprehensively as we did in [Example 1](#); and second, we were left with having to integrate $\sec u$, which we did by quoting its integral as a standard result – this result of this integral is quite complicated in itself.

As we noted earlier with our example of sine-substitution, the reasoning we used in [Example 2](#) can be used to evaluate any integral of the form:

$$I = \int \frac{1}{\sqrt{(x^2 + a^2)}} dx$$

Example 3

Our third example of trigonometric substitution will be the following integral:

$$I = \int \frac{1}{\sqrt{(x^2 - 4)}} dx$$

Once again, we want to use our new trigonometric identity here, but this time in a different rearrangement. We will therefore try the substitution $x = 2 \sec u$.

With this substitution, we find:

$$x^2 - 4 = 4\sec^2 u - 4 = 4(\sec^2 u - 1) = 4\tan^2 u$$

using the second rearrangement of our trigonometric identity. And, of course, it will come as no surprise to hear that this approach is sometimes called “secant-substitution”.

Under substitution, the denominator of our integrand becomes:

$$\sqrt{x^2 - 4} = \sqrt{4\tan^2 u} = 2 \tan u$$

Integration by Substitution

In order to obtain an expression for dx , we proceed as follows:

$$\frac{dx}{du} = 2 \sec u \tan u$$

so that $dx = 2 \sec u \tan u \, du$.

Substituting all this back into our original integral, we are left with:

$$\int \frac{2 \sec u \tan u}{2 \tan u} du = \int \sec u \, du = \ln|\sec u + \tan u| + C$$

In order to reverse the substitution, we first obtain an expression for u in terms of x , and we are left with the following result:

$$\ln \left| \frac{1}{2}x + \tan \sec^{-1} \frac{1}{2}x \right| + C$$

obtaining the first, simplified, term inside the modulus brackets by keeping in mind that $\sec \sec^{-1} a = a$.

The second term can be simplified in a similar way to the way we dealt with the result of [Example 2](#). In this case, we want to simplify the expression $\tan \sec^{-1} x$. Once again, there is a convenient identity which will help us – listed and derived in [Appendix 4 Section 4](#) – this identity is:

$$\tan \sec^{-1} x = \sqrt{x^2 - 1}$$

This gives us another simplification of the combination of a trigonometric function and an inverse trigonometric function which is completely free of trigonometric functions. We can use this to rewrite our earlier solution to the integral of the current example as follows:

$$\ln \left| \frac{1}{2}x + \sqrt{\frac{1}{4}x^2 - 1} \right| + C$$

As in [Example 2](#), we can aspire to do even better: we can rewrite the expression inside the square root as follows:

$$\frac{1}{4}x^2 - 1 = \frac{x^2 - 4}{4}$$

The square root of this expression, when we take the denominator outside the square root sign, becomes:

$$\frac{1}{2}\sqrt{x^2 - 4}$$

This turns our solution into the following expression:

Integration by Substitution

$$\ln \left| \frac{1}{2}(x + \sqrt{x^2 - 4}) \right| + C$$

Using the rules of logarithms, as we did in the previous example, this becomes:

$$\ln \left(\frac{1}{2} \right) + \ln |x + \sqrt{x^2 - 4}| + C$$

Finally, restating our constant of integration as follows:

$$C' = C + \ln \left(\frac{1}{2} \right)$$

we arrive at a final solution, expressed in similar terms to those of the integrand, as follows:

$$I = \ln |x + \sqrt{x^2 - 4}| + C'$$

Note

Although we concluded the central work of integration in this example by integrating the same function of u as we did in [Example 2](#), the final result is not quite as big a monster as we obtained in [Example 2](#), due to the absence of square roots – we started with a constant term of 4, which is a perfect square. However, its multiple layers still bear witness to the greater complexity produced by tangent- and secant-substitution as compared with sine-substitution.

We can round off this set of examples by pointing out that the same reasoning as we used here can be used to evaluate any integral of the form:

$$I = \int \frac{1}{\sqrt{(x^2 - a^2)}} dx$$

For the greater part of this section, we have been looking at integrals with square roots in the denominator of a rational function. It would be a mistake to give the impression that trigonometric substitution is limited to such cases; and in order to make this clear, the next example in this section will illustrate an application of trigonometric substitution to a case where a square root of the kind we have been looking at is present on its own, and not as part of a rational function.

Example 4

Here is our integral:

$$I = \int \sqrt{16 - x^2} dx$$

This looks not unlike the first example we worked through in this section. In that earlier case, of course, the square root was in the denominator of a fraction, and we were able to do some comprehensive cancelling once we had made our substitution. Such cancellation won't

be available to us here; but let's try a trigonometric substitution anyway, and follow where that leads.

The expression under the square root is of the type which might attract us to the idea of a sine-substitution, so we will try $x = 4 \sin u$. Using this substitution, we will find that:

$$16 - x^2 = 16 - 16\sin^2 u = 16(1 - \sin^2 u) = 16\cos^2 u$$

Proceeding further, as we have done in earlier examples

$$\sqrt{16 - x^2} = \sqrt{16\cos^2 u} = 4 \cos u$$

and in order to find an expression for dx :

$$\frac{dx}{du} = 4 \cos u$$

so that $dx = 4 \cos u \, du$.

All this transforms our integral into:

$$\int 16\cos^2 u \, du$$

We note once more, as we did a little earlier, that we have not been able to carry out any cancelling between a numerator and a denominator, but we are instead left with the square of a trigonometric function, which at first sight doesn't make things any easier.

We can't argue that perhaps we have chosen a wrong substitution, however, since all the x terms in the transformed integral have obligingly disappeared. And indeed, there is a step we can take at this point in order to allow us to carry on with the integration – we can use the “power-reduction formula”:

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

(This formula can be derived from the “addition formula” for $\cos(\alpha + \beta)$ by setting $\alpha = \beta = x$.)

Using this identity, we find that our integral has been transformed into the rather easier:

$$\int 8(1 + \cos 2u)du = \int (8 + 8 \cos 2u)du$$

We can now integrate this very simply if we are prepared to use the rule for integrating expressions of the form $\cos kx$ which is quoted in [Appendix 4 Section 2](#).

Our result in terms of u is:

$$8u + 4 \sin 2u + C$$

Integration by Substitution

However, this is still not a result in terms of x : in order to achieve that, we have to obtain an expression from our substitution, which was a trigonometric one in the “reversed” notation we have used for these. An expression for u in terms of x is required, and our original substitution expression can be rearranged to give the following:

$$u = \sin^{-1} \frac{1}{4}x$$

Putting this into the solution in terms of u which we obtained above, we are left with:

$$8 \sin^{-1} \frac{1}{4}x + 4 \sin(2 \sin^{-1} \frac{1}{4}x) + C$$

This is not the prettiest result in the world, but at least it’s a result. We can try to make something a little less ugly out of it by noting that the second term is of the form $4 \sin 2A$, so that we can apply the double-angle formula:

$$\sin 2x = 2 \sin x \cos x$$

This transforms the term in question into:

$$8 \sin \sin^{-1} \frac{1}{4}x \cos \sin^{-1} \frac{1}{4}x = 2x \cos \sin^{-1} \frac{1}{4}x$$

which is slightly less inelegant.

In line with the work we did at the end of the previous two examples, we might now ask whether we can simplify the expression of the form $\cos \sin^{-1} x$ which makes its appearance here. Indeed there is a trigonometric identity which does this very job. It is as follows:

$$\cos \sin^{-1} x = \sqrt{1 - x^2}$$

and its derivation is given along with other similar derivations in [Appendix 4 Section 4](#).

We can now make use of this identity and bring our final solution into the form

$$8 \sin^{-1} \frac{1}{4}x + 2x \sqrt{1 - \frac{1}{16}x^2} + C$$

The expression inside the square root sign can be rewritten and generally improved as follows:

$$1 - \frac{x^2}{16} = \frac{16 - x^2}{16}$$

Taking the denominator of 16 outside the square root sign turns the second term of our solution into:

Integration by Substitution

$$\frac{2x}{\sqrt{16}}\sqrt{16-x^2} = \frac{2x}{4}\sqrt{16-x^2} = \frac{1}{2}x\sqrt{16-x^2}$$

leaving us with the following final solution:

$$I = 8 \sin^{-1} \frac{1}{4}x + \frac{1}{2}x\sqrt{16-x^2} + C$$

The moral of this example, if it is legitimate to draw one, must be this: you can't afford to be squeamish about a little algebra or trigonometry when doing calculus.

Note

This last example serves to indicate that the method of trigonometric substitution can work well on a range of integrands which contain square roots of the kind encountered in the examples in this section. These have been of the following forms:

$$\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2} \text{ or } \sqrt{x^2 - a^2}$$

Accordingly, we might expect to find that any integrand containing one of these square root expressions might be a promising candidate for trigonometric substitution. The exact substitution we should choose, of course, varies depending on which of the above expressions is present.

We can summarise the choices we have made in this section using the following table:

Square Root Expression	Recommended Substitution
$\sqrt{a^2 - x^2}$	Sine-substitution
$\sqrt{x^2 + a^2}$	Tangent-substitution
$\sqrt{x^2 - a^2}$	Secant-substitution

A number of useful standard integrals arise from the use of trigonometric substitution – some of these have been mentioned in the various notes following the examples of this section. A fuller list is given for ease of reference in [Appendix 4 Section 2](#).

Example 5

We will conclude this section with an example of a more sophisticated application of trigonometric substitution, which will use the properties of the trigonometric functions to reduce a difficult integral to a form where it can be easily dealt with. Our example will be the following integral:

$$I = \int \frac{1}{\sqrt{x(1-x)}} dx$$

We immediately note that there are no explicit squared terms in the denominator (unless, of course, we multiply out the expression under the square root sign – even then, we are left with one term which is not squared).

Integration by Substitution

In order to open up the possibility of using the Pythagorean identity, we will try introducing some squared terms by suggesting the following substitution:

$$x = \sin^2 u$$

This is unlike any of the sine-substitutions we have used so far: however, let's list its consequences, and see if it will help us. We first have by means of the Chain Rule for differentiation:

$$\frac{dx}{du} = 2 \sin u \cos u \text{ so that } dx = 2 \sin u \cos u du$$

We further note that the denominator of our integrand is transformed as follows:

$$\sqrt{\sin^2 u (1 - \sin^2 u)} = \sqrt{\sin^2 u \cos^2 u} = \sin u \cos u$$

Putting all this together, we will find a surprisingly convenient integral in terms of u , which will be as follows:

$$\int \frac{1}{\sin u \cos u} 2 \sin u \cos u du = \int 2 du$$

This, of course, integrates to give the solution:

$$2u + C$$

All that is left is to reverse the substitution. This requires us to obtain an expression for u in terms of x . Taking the square root of both sides of our original substitution formula gives us:

$$\sin u = \sqrt{x} \text{ so that } u = \sin^{-1} \sqrt{x}$$

From this result, we can now write down our solution in terms of x as follows:

$$I = 2 \sin^{-1} \sqrt{x} + C$$

This is, of course, a rather exotic solution, combining as it does an inverse sine function with a square root. And indeed, we ought to remark that the integrand was actually in just the right shape for the substitution we chose to work and allow us to cancel almost everything out when obtaining our integral in terms of u . This example, then, is something of a “just-so story”, although it may serve to illustrate how a cunning choice of trigonometric substitution can be of great help in dealing with awkward integrals – particularly where the presence of a square root indicates that a useful rule such as the trigonometric Pythagorean identity might make things much easier for us.

12 Integration in Several Stages

In the last few sections, we have seen how different forms of the substitution technique can be used in conjunction with simple algebra or the use of trigonometric identities to evaluate

integrals. We will conclude this survey of integration by substitution with a look at a few examples of the successive application of several techniques, including algebra, trigonometry and even the use of more than one substitution, in order to evaluate integrals which do not at first look like promising candidates for our technique. This will give some idea of the true scope of the method's applicability.

Example 1

For my first example of multi-stage integration, using both successive substitutions and some algebra, I am going to consider an example I found in my lecture notes from university. I will, however, make every effort to give a much more detailed explanation of the integration process than was provided in my notes, or indeed in the lecture at which they were taken.

The indefinite integral we are asked to consider is as follows:

$$I = \int \frac{1}{\sqrt{4 - 2x - x^2}} dx$$

Based on our experience of the preceding examples, we might think that a substitution such as $u = 4 - 2x - x^2$, where we substitute the entire quadratic expression under the square root, might be a good idea. However, when we find the corresponding expression for dx , it becomes obvious that this approach leaves us with unresolved terms in x , so in accordance with our algorithm for integration by substitution, we must go back and think again.

At this point, we might consider an algebraic technique which can be applied to quadratic expressions to render them more tractable in a number of ways: this is the technique known as *completing the square*. Since this technique can be very useful in getting an integrand containing a quadratic into a form where we can make a useful substitution, I give a full account of how to use the technique in a range of situations in [Appendix 3 Section 1](#). For the sake of this example, however, I will go through the process in a more informal way.

Looking at the quadratic under the square root, we first notice that the order of terms is reversed from the more common one, and in addition, the signs of the terms in x and x^2 are both negative. In order to approach the form which we may be more used to, we can choose to rewrite the quadratic as follows:

$$4 - (x^2 + 2x)$$

Now that we have it in this form, we might note that the part in brackets only lacks a constant term of 1 to become the expansion of $(x + 1)^2$. And when we reflect that an expression of the form $a - b$ retains the same value if we simply add 1 to each term, obtaining the result $(a + 1) - (b + 1)$, we should readily agree that our rearranged quadratic can be rewritten as:

$$(4 + 1) - (x^2 + 2x + 1) = 5 - (x + 1)^2$$

Our indefinite integral now therefore looks like this:

$$\int \frac{1}{\sqrt{5 - (x + 1)^2}} dx$$

This immediately seems a little friendlier towards the substitution technique, indeed the integrand could be regarded as a function of the form $f(ax + b)$, which as we know should respond to the substitution $u = ax + b$. We will therefore try $u = x + 1$, which as we have seen above gives us the simple equivalence $dx = du$. This gives us the new integral:

$$\int \frac{1}{\sqrt{5 - u^2}} du$$

We should immediately be overcome by the sensation that we have seen something very like this before. And indeed we have, in [Example 1 of Section 11](#) above, where we used sine-substitution. Of course, 5 is not a perfect square, as the constant 9 was in the above example, but we should by now not be deterred by a mere square root. Accordingly, we will make the *further* (trigonometric) substitution $u = \sqrt{5} \sin v$. (Note that we require the additional substitution variable v this time, since u has already been used once!)

We will now proceed to repeat the steps we took in [Example 1 of Section 11](#). Using our sine-substitution, we find that the expression under the square root becomes:

$$5 - u^2 = 5 - 5\sin^2 v = 5(1 - \sin^2 v) = 5\cos^2 v$$

As an immediate consequence, we can say:

$$\sqrt{5 - u^2} = \sqrt{5\cos^2 v} = \sqrt{5} \cos v$$

To obtain an expression for du , we state that:

$$\frac{du}{dv} = \sqrt{5} \cos v \text{ so that } du = \sqrt{5} \cos v dv$$

Putting these conclusions back into our integral in u , it becomes:

$$\int \frac{\sqrt{5} \cos v}{\sqrt{5} \cos v} dv = \int 1 dv = v + C$$

This is the agreeably simple result we reached at the end of [Example 1 of Section 11](#). As in that case, reversing the trigonometric substitution gives the result:

$$\sin^{-1} \frac{1}{\sqrt{5}} u + C$$

and with one final piece of substitution-reversing, we obtain the following result in terms of x :

$$I = \sin^{-1} \frac{1}{\sqrt{5}} (x + 1) + C$$

An Alternative Approach

Having reached our solution using the techniques we have already discussed, this would be a good stage to point out that the second substitution we used to solve the above integral could be replaced by an algebraic substitution which would get us to the same result by a different route. Here is the reasoning.

Starting with the integral in terms of u :

$$\int \frac{1}{\sqrt{5-u^2}} du$$

We will turn our backs on the simple route provided by sine-substitution, and instead look for something algebraic which might simplify our integrand. We will also turn our backs on the possibility of power-substitution, as explained in [Section 8](#) above, and instead try the algebraic substitution $u = \sqrt{5}v$, reflecting for a moment that it would probably take some trial and error to arrive at this substitution from scratch. We note that this gives us:

$$\frac{du}{dv} = \sqrt{5}$$

so that $du = \sqrt{5}dv$.

The integral in u now turns into the following:

$$\int \frac{\sqrt{5}}{\sqrt{5-5v^2}} dv = \int \frac{\sqrt{5}}{\sqrt{5}\sqrt{1-v^2}} dv = \int \frac{1}{\sqrt{1-v^2}} dv = \sin^{-1} v + C$$

treating the result we demonstrated in the note following [Example 1 of Section 11](#) above as a standard integral.

Reversing each of our substitutions in turn gives us the same successive results we presented at the end of the main discussion of this example, and we are, of course, left after reversing both our substitutions with exactly the same final result in terms of x , namely:

$$I = \sin^{-1} \frac{1}{\sqrt{5}}(x+1) + C$$

Example 2

For our second example, we will consider an integral which is a little more complicated than any we have so far examined. The integral we will attempt to evaluate is as follows:

$$I = \int \frac{x}{x^2 + 2x + 2} dx$$

Learning from the approach which worked in [Example 1](#) above, we might start by completing the square for the quadratic in the denominator. It should be easy to see, even without

adopting a formal method for completing the square, that this quadratic is simply the expansion of $(x + 1)^2$ with a constant term of 1 added.

We can therefore rewrite our integral as:

$$\int \frac{x}{(x + 1)^2 + 1} dx$$

This almost cries out to be treated using the substitution $u = x + 1$, which of course gives us the very convenient equivalence $dx = du$, as we saw in [Example 1](#) above. Our integral is now transformed into the following form:

$$\int \frac{u - 1}{u^2 + 1} du$$

At first sight, this is really no less awkward than the integral in x with which we started out. However, not to be deterred, let's see if a little simple algebra can be brought into play to help us out. And indeed one simple idea might be to split this single fraction in terms of u into two separate fractions, each of which we can then integrate separately. And if we do this, we quickly find that the two integrals we obtain look a tiny bit like some we have dealt with before.

Dividing the integrand into two separate fractions with the same denominator, therefore, we now have to integrate:

$$\int \frac{u}{u^2 + 1} du - \int \frac{1}{u^2 + 1} du$$

If we now name these integrals I_1 and I_2 , we can state that the solution to our original integral will simply be the difference of the two. In other words:

$$I = I_1 - I_2$$

Let's now deal with each of the two integrals into which we have decomposed I as two separate sub-problems. We'll put the results back together again at the end of our discussion.

Example 2: Evaluation of I_1

Let's start with the first of the two integrals, which is as follows:

$$I_1 = \int \frac{u}{u^2 + 1} du$$

We might at first lean towards a trigonometric substitution here. However, noting the u term in the numerator should give us an important clue on how best to approach this integral. In fact, the integrand here is of the form we set out in [Section 4](#) above as being suitable for the basic substitution approach. The integrand is of the form:

$$kg'(f(u))f'(u)$$

where:

$$g'(f(u)) = \frac{1}{u^2 + 1}; \quad f(u) = u^2 + 1; \quad f'(u) = 2u; \quad k = \frac{1}{2}$$

This, of course, makes straightforward u -substitution available to us (although we will, of course have to use v rather than u since the integral is already expressed in terms of u), and so we can use the substitution $v = u^2 + 1$, substituting for the entire denominator. From this we will obtain, in line with the structure of the integrand as just set out:

$$\frac{dv}{du} = 2u \text{ so that } du = \frac{1}{2u} dv$$

We immediately note that the u term in the denominator of the expression for du will now cancel with the u term in the numerator of the integrand, so that our integral becomes:

$$\int \frac{u}{v} \cdot \frac{1}{2u} dv = \int \frac{1}{2v} dv = \int \frac{1}{2} \cdot \frac{1}{v} dv = \frac{1}{2} \int \frac{1}{v} dv$$

I have taken the coefficient of $1/2$ outside the integral sign to draw out the fact that we can now use the standard integral:

$$\int \frac{1}{x} dx = \ln|x| + C$$

This allows us to evaluate our integral in v to arrive at the following result:

$$\frac{1}{2} \ln v + C = \frac{1}{2} \ln(u^2 + 1) + C$$

The final result above is, of course, obtained by reversing the v -substitution to obtain a result in terms of u .

Example 2: Evaluation of I_2

We can now turn our attention to the second of our two integrals, which is as follows:

$$I_2 = \int \frac{1}{u^2 + 1} du$$

Recalling the work we did in [Section 9 Example 2](#), there could be a suggestive case here for tangent-substitution. We therefore waded boldly in and consider the substitution $u = \tan v$.

This substitution has the following effect on our denominator:

$$u^2 + 1 = \tan^2 v + 1 = \sec^2 v$$

As for our expression for dv , we proceed as follows:

Integration by Substitution

$$\frac{du}{dv} = \sec^2 v \text{ so that } du = \sec^2 v \, dv$$

Putting these conclusions back into our integral, we obtain:

$$\int \frac{\sec^2 v}{\sec^2 v} dv = \int 1 \, dv = v + C$$

(We can see that this is an even neater result than the one we obtained when evaluating the integral in [Section 11 Example 2](#), which had a square root in its denominator.)

The final step for this integral is to reverse the substitution, obtaining a result in terms of u - and this is, very simply:

$$\tan^{-1} u + C$$

(Let's take a moment to emphasise this conclusion - of which you may already have been aware - that we have actually demonstrated another standard integral, which is frequently used without detailed justification. That standard integral is:

$$\int \frac{1}{x^2 + 1} dx = \tan^{-1} x + C$$

This is a similar result to the one we remarked upon at the end of the discussion of [Section 11 Example 1](#).)

Completing Example 2

We can now remind ourselves that we have divided our integral in terms of u into two separate integrals, I_1 and I_2 , which must be put back together to give us our final result using the relation:

$$I = I_1 - I_2$$

Taking our results in terms of u from the two previous sections, then, we arrive at the following position:

$$I = \frac{1}{2} \ln(u^2 + 1) - \tan^{-1} u + C$$

Now all that remains is to reverse our u -substitution in order to achieve a solution in terms of x . Recalling that we originally made the substitution $u = x + 1$, we can transform the solution very simply into the following:

$$\frac{1}{2} \ln((x + 1)^2 + 1) - \tan^{-1}(x + 1) + C$$

Taking the final step of expanding out the $(x + 1)^2$ term, we get a final solution which is satisfyingly reminiscent of the structure of the original integrand:

Integration by Substitution

$$\frac{1}{2} \ln(x^2 + 2x + 2) - \tan^{-1}(x + 1) + C$$

An Interesting Curiosity

In the university notes from which I drew this example, we were actually asked to evaluate the *definite* integral:

$$\int_0^1 \frac{x}{x^2 + 2x + 2} dx$$

Having evaluated the corresponding *indefinite* integral just a moment ago, we can immediately write down the solution to the definite integral as:

$$\left[\frac{1}{2} \ln(x^2 + 2x + 2) - \tan^{-1}(x + 1) \right]_0^1$$

By putting into place the upper and lower limits of integration, this evaluates to:

$$\frac{1}{2} \ln 5 - \tan^{-1} 2 - \frac{1}{2} \ln 2 + \tan^{-1} 1$$

We can simplify this by noting that:

$$\tan \frac{\pi}{4} = 1 \text{ so that } \tan^{-1} 1 = \frac{\pi}{4}$$

and additionally making use of the following property of logarithms:

$$\ln a - \ln b = \ln \left(\frac{a}{b} \right)$$

With the aid of these two identities, our solution now simplifies to the following form:

$$\frac{\pi}{4} + \frac{1}{2} \ln \left(\frac{5}{2} \right) - \tan^{-1} 2$$

We therefore see that the area under the curve:

$$y = \frac{x}{x^2 + 2x + 2}$$

between the lines $x = 0$ and $x = 1$ is obtained by combining a fraction of the transcendental number π , a logarithm to the transcendental base e and an inverse trigonometric function. It almost makes one wonder how the areas under some other curves turn out to be such simple quantities as they do.

Example 3

We conclude this section with one further example which will serve to show how wide the applicability of the approach we have been using here really is. Our integral will be less complicated in its initial appearance than the two preceding ones, but it will still require more than one application of substitution.

We will evaluate the following integral:

$$I = \int \sqrt{4 - 2x - x^2} dx$$

The integrand here is, of course, none other than the square root expression which comprised the denominator of the integrand in [Example 1](#) above. As well as giving us a slightly different kind of integrand for the application of the methods we are using in this section, it will perhaps also prove interesting to see how the solution compares to the solution we obtained when we integrated the reciprocal of the current expression in [Example 1](#).

We have not had to deal with an integral looking quite like this one before in the course of this book. However, we might begin by recalling how we decided to treat the square root expression when it appeared in the denominator of the integrand in [Example 1](#) above. There, we completed the square for the quadratic in question, and that opened the way for a simple substitution as the first step towards a solution.

Following this line of attack then, we can begin by simply quoting some of the results we derived at the start of [Example 1](#).

First, completing the square for the quadratic under the square root sign gives us this equivalence:

$$4 - 2x - x^2 = 5 - (x + 1)^2$$

which transforms our integral into:

$$\int \sqrt{5 - (x + 1)^2} dx$$

As we did with the first substitution in [Example 1](#), we will again choose $u = x + 1$, giving us the equivalence $dx = du$. We now have the following integral in terms of u :

$$\int \sqrt{5 - u^2} du$$

The arguments offered in [Section 11](#) above, not to mention the more recent demonstration in [Example 1](#) of this section, should point us decisively towards sine-substitution – the very same sine-substitution which worked so well in [Example 1](#), in fact.

We will recall that it was:

Integration by Substitution

$$u = \sqrt{5} \sin v \text{ so that } \frac{du}{dv} = \sqrt{5} \cos v \text{ and } du = \sqrt{5} \cos v dv$$

Once again, the expression under the square root becomes:

$$5 - 5\sin^2 v = 5(1 - \sin^2 v) = 5\cos^2 v$$

and we arrive at the following integral in terms of v :

$$\int \sqrt{5\cos^2 v} \cdot \sqrt{5} \cos v dv = \int \sqrt{5} \cos v \cdot \sqrt{5} \cos v dv = 5 \int \cos^2 v dv$$

For the final right-hand side, we have multiplied together the two instances of $\sqrt{5}$ in the preceding version of the integrand and taken the resulting coefficient of 5 outside the integral sign.

We can integrate the final right-hand side above fairly easily by applying the trigonometric power-reduction formula for the cosine – this is among the trigonometric formulae listed in [Appendix 4 Section 3](#), and it is as follows:

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Putting this formula to work on our integral in terms of v , we find that it becomes the following:

$$\frac{5}{2} \int (1 + \cos 2v) dv = \frac{5}{2} \left(v + \frac{1}{2} \sin 2v \right) + C$$

We are now at the stage where we can begin reversing substitutions so as to reach our final solution. First, we must recall that the trigonometric substitution which produced our integral in terms of v was as follows:

$$u = \sqrt{5} \sin v \text{ so that } \sin v = \frac{1}{\sqrt{5}} u \text{ and finally } v = \sin^{-1} \frac{1}{\sqrt{5}} u$$

This formula can be inserted into our earlier solution in terms of v to obtain the following solution in terms of u :

$$\frac{5}{2} \left(\sin^{-1} \frac{1}{\sqrt{5}} u + \frac{1}{2} \sin 2 \sin^{-1} \frac{1}{\sqrt{5}} u \right) + C$$

While this is undoubtedly a solution of sorts, the combination of the sine and inverse sine functions is hardly the most elegant thing in the world, and we might expect to be able to reduce it to something rather less unwieldy.

To help us improve on this clumsy expression, we can call to our assistance the double-angle formula for the sine – again, this can be found listed in [Appendix 4 Section 3](#), and I quote it here for those who may not be familiar with it:

Integration by Substitution

$$\sin 2x = 2 \sin x \cos x$$

In our present case, we apply this formula with the substitution:

$$x = \sin^{-1} \frac{1}{\sqrt{5}} u$$

This lets us rewrite our undesirable expression as follows:

$$\sin 2 \sin^{-1} \frac{1}{\sqrt{5}} u = 2 \left(\sin \sin^{-1} \frac{1}{\sqrt{5}} u \right) \left(\cos \sin^{-1} \frac{1}{\sqrt{5}} u \right)$$

The two bracketed factors on the right-hand side can now be dealt with separately. First, we know that:

$$\sin \sin^{-1} x = x \text{ so that } \sin \sin^{-1} \frac{1}{\sqrt{5}} u = \frac{1}{\sqrt{5}} u$$

In the case of the second factor, we can quote one of the formulae for combinations of trigonometric and inverse trigonometric functions which are listed and derived in [Appendix 4 Section 4](#). The one we need in this case is:

$$\cos \sin^{-1} x = \sqrt{1 - x^2} \text{ so that } \cos \sin^{-1} \frac{1}{\sqrt{5}} u = \sqrt{1 - \frac{1}{5} u^2}$$

The expression under the square root can be rewritten to make it simpler to handle as follows:

$$1 - \frac{1}{5} u^2 = 1 - \frac{u^2}{5} = \frac{5 - u^2}{5}$$

This lets us restate the above simplification of the cosine and inverse sine combination as follows:

$$\cos \sin^{-1} \frac{1}{\sqrt{5}} u = \sqrt{\frac{5 - u^2}{5}} = \frac{1}{\sqrt{5}} \sqrt{5 - u^2}$$

Assembling all this together, we can now simplify our awkward expression as follows:

$$\sin 2 \sin^{-1} \frac{1}{\sqrt{5}} u = 2 \cdot \frac{1}{\sqrt{5}} u \cdot \frac{1}{\sqrt{5}} \sqrt{5 - u^2} = \frac{2}{5} u \sqrt{5 - u^2}$$

The solution in terms of u can now be written out more elegantly as follows:

$$\frac{5}{2} \left(\sin^{-1} \frac{1}{\sqrt{5}} u + \frac{1}{2} \cdot \frac{2}{5} u \sqrt{5 - u^2} \right) + C = \frac{5}{2} \sin^{-1} \frac{1}{\sqrt{5}} u + \frac{1}{2} u \sqrt{5 - u^2} + C$$

Integration by Substitution

All we have to do now is to reverse the opening substitution, which was $u = x + 1$. Simply substituting $x + 1$ for each instance of u produces this expression:

$$\frac{5}{2} \sin^{-1} \frac{1}{\sqrt{5}}(x+1) + \frac{1}{2}(x+1)\sqrt{5 - (x+1)^2} + C$$

Rather than leaving it at that, we ought to recall that our very first step – completing the square for the quadratic under the square root in the original integrand – gave us this equivalence:

$$4 - 2x - x^2 = 5 - (x + 1)^2$$

We therefore ought to replace the expression under the square root in the above solution in x with the quadratic on the left-hand side above, in order to bring the solution into line as far as we can with the original integrand. Taking this, perhaps largely cosmetic, step will produce our final solution of:

$$I = \frac{5}{2} \sin^{-1} \frac{1}{\sqrt{5}}(x+1) + \frac{1}{2}(x+1)\sqrt{4 - 2x - x^2} + C$$

Out of interest, let's conclude by comparing this with the solution we obtained for the integral of the reciprocal of the integrand in this example. That solution, obtained in [Example 1](#) above, was:

$$\int \frac{1}{\sqrt{4 - 2x - x^2}} dx = \sin^{-1} \frac{1}{\sqrt{5}}(x+1) + C$$

Fascinatingly, we see that the solution in the present case contains the solution from [Example 1](#) multiplied by a coefficient of $5/2$, with the addition of an extra term containing the original integrand as a factor.

Note

Consideration of these three examples allows us to see that we can now describe a further class of integrals which are amenable to substitution – admittedly with the aid of some algebra and trigonometry at various stages, and by means of more than one successive substitution. This class includes integrals of the following three forms:

$$\int \frac{Ax + B}{\sqrt{Cx^2 + Dx + E}} dx$$

also

$$\int \frac{Ax + B}{Cx^2 + Dx + E} dx$$

and:

Integration by Substitution

$$\int \sqrt{Ax^2 + Bx + C} dx$$

These integrals require the following techniques: first, completing the square to put the quadratic expression into a form where a substitution becomes obvious; then two successive substitutions, one of which may be a trigonometric substitution if the first substitution brings us an integral in one of the forms we explored in [Section 11](#). In cases where the coefficient A in the numerator is non-zero (as we saw in [Example 2](#) above), we will find that after the first substitution, it will be necessary to divide our integrand into two separate fractions, using the simple algebraic relation:

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

Each of the resulting fractions can then be integrated separately. Each one will typically require a different kind of substitution.

Finally, it is useful to remember, as we saw in the alternative approach to [Example 1](#), that if we are prepared to quote as standard integrals the expressions which integrate to the various inverse trigonometric functions, we may be able to use an algebraic substitution as the second substitution in the above algorithm instead of a trigonometric one.

Part 2

Integration by Parts

1 Substitution is Not Always the Answer

Let's start our examination of a second useful technique for integration by considering another "difficult" function which we cannot readily integrate using our basic rules:

$$I = \int x \cos x \, dx$$

The integrand here lacks the symmetry of the one we considered at the start of [Part 1](#) which allowed the expression for dx which we derived as part of the substitution technique to be used to cancel out a part of it. This symmetry reflected the symmetry of the basic trigonometric functions under differentiation.

We should note right at the outset that this integral does not conform to the structure we established in [Part 1](#) for the most basic use of substitution, but since that hasn't stopped us before, let's see if substitution can get us anywhere. Leaving aside any consideration of algebraic or trigonometric substitutions – anything we might try simply seems to complicate rather than simplify the integrand – we probably only have two real choices for u -substitution.

One of these, the trivial substitution $u = x$, gets us precisely nowhere since it doesn't alter the integrand at all. The second option, which would be $u = \cos x$, gives us the following:

$$\frac{du}{dx} = -\sin x \text{ so that } dx = -\frac{1}{\sin x} du$$

This introduces an additional term of $\sin x$ in the denominator of a fraction, which cannot be cancelled with anything in the original integrand. Indeed, it turns the integral into:

$$-\int \frac{u \cos^{-1} u}{\sin x} du$$

This contains an unresolved term in x ; is there anything we can do about this? Well, one course opens up to us if we remember that there is a relationship between $\sin x$ and $\cos x$ given by the Pythagorean identity:

$$\sin^2 x + \cos^2 x = 1$$

This allows us, by rearrangement, to say:

$$\sin x = \sqrt{1 - \cos^2 x}$$

Since the substitution we chose to try in this case was $u = \cos x$, this means that we can rewrite the denominator of our transformed integral as:

$$\sqrt{1 - u^2}$$

So that the integral as a whole is transformed into:

$$- \int \frac{u \cos^{-1} u}{\sqrt{1 - u^2}} du$$

Now, at least this doesn't leave any unresolved terms in x lying around, so it doesn't fail the first test in our algorithm for integration by substitution. However, it has not made the integral any simpler, which is what we look for when we use substitution. Indeed, it has made it considerably uglier and more disturbing.

So, barring our invention of any special algebraic tricks like that used in [Appendix 5 Section 6](#) for the integration of $\sec x$, or our preparedness to try applying substitution again to see if that will help, we are forced to the conclusion that if we are going to evaluate this integral in a simple fashion, we will need a new kind of technique altogether.

2 The Product Rule for Differentiation and its Consequences

In order to introduce a new technique which we will be able to use to evaluate our example integral, we need to start by renewing our acquaintance with the Product Rule, which allows us to write down with ease the derivative of any function which has the form $f(x)g(x)$, or in other words, is formed of the product of two simpler functions.

The Product Rule for differentiating such functions states that, if $F(x) = f(x)g(x)$, then:

$$F'(x) = f'(x)g(x) + f(x)g'(x)$$

This is frequently written differently, by setting the variables $u = f(x)$ and $v = g(x)$ and using Leibniz notation, to give the following restatement:

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

I will stick with this second version of the rule for the moment, since it allows a neater statement of the consequence I want to derive. This has also been a common presentation of the rule in textbooks. It is worth pointing out that a shorthand form of this version using the “dashed” notation is sometimes given, which you may find easier to remember, and which looks like this:

$$(uv)' = u'v + uv'$$

We will start by integrating each side of the full-blown Leibniz form of our equation with respect to x . This will give us the new equality:

Integration by Parts

$$uv = \int \frac{du}{dx} v dx + \int u \frac{dv}{dx} dx$$

A minor rearrangement, so as to make the second term on the right-hand side the subject of the equation, leads to this interesting result:

$$\int u \frac{dv}{dx} dx = uv - \int \frac{du}{dx} v dx$$

In order to give the most complete presentation of this result, let's take a look at a version of it using function notation. Reversing the substitution of the variables u and v which we described above, that would run as follows:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Now, from the point of view of evaluating integrals, this is a potentially promising conclusion to have reached. It means that if we are asked to evaluate an integral which has the form:

$$\int f(x)g'(x)dx$$

ie an integral comprising a product of two functions, one of which can be identified as a derivative, then this integral can be expressed as the difference

$$f(x)g(x) - \int f'(x)g(x)dx$$

This will be a useful representation if the following two conditions are met: first, it needs to be fairly easy to get from $g'(x)$ to $g(x)$ - in other words, $g'(x)$ needs to be a function which is easy to integrate; and second, the integral:

$$\int f'(x)g(x)dx$$

must be easier to evaluate than the integral we started out with, that is:

$$\int f(x)g'(x)dx$$

In cases where these two conditions are met, we have in this equation a potentially useful new approach to the evaluation of difficult integrals.

You may encounter in some textbooks a version of the u - v expression of our new result which uses the “differentials” du and dv . I don't intend to use such a version in this book, since it seems to be a little opaque to anyone not thoroughly used to that kind of notation and able to interpret it clearly. I do, however, give an account of this version in [Appendix 4 Section 5](#) for the sake of completeness.

3 Integration by Parts

We will now try to evaluate the integral with which we opened this part of the book using the interesting result we derived in the previous section.

By way of reminder, the integral we want to evaluate is as follows:

$$I = \int x \cos x \, dx$$

and the useful formula which we hope will help us to integrate it is as follows:

$$\int u \frac{dv}{dx} dx = uv - \int \frac{du}{dx} v dx$$

In order to use this formula, we will have to allocate the two parts of the integral on the left-hand side of the formula to the two factors in the product of functions in our integral. As a start, let's see what happens if we set:

$$u = x \text{ and } \frac{dv}{dx} = \cos x$$

We can immediately see that these choices give us the following consequences:

$$\frac{du}{dx} = 1 \text{ and } v = \sin x$$

Substituting all the values for u , v and their derivatives into the right-hand side of our useful formula, we get the result:

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C$$

In case you may be unprepared to believe this result, we can check it by differentiating the final expression above. This requires an application of the Product Rule in order to differentiate the first term, and gives us the following:

$$\sin x + x \cos x - \sin x = x \cos x$$

which gets us back to the original integrand since the two terms in $\sin x$ cancel each other out.

The application of our formula, with the integrand we are given being divided into two parts which we can represent using the variables u and v (along with the derivatives of the functions in x which they represent), is called *integration by parts*. This is the second of the techniques for integration which this book sets out to explain

Before leaving the integral we have been examining, it will be instructive to see what happens if we allocate u and v in a different way from the choice we made above. The only other alternative would be the following:

Integration by Parts

$$u = \cos x \text{ and } \frac{dv}{dx} = x$$

The consequences of these choices are as follows:

$$\frac{du}{dx} = -\sin x \text{ and } v = \frac{1}{2}x^2$$

Things are already beginning to look a little more complicated than they did with our earlier set of choices, so let's see what happens when we plug the above expressions into the right-hand side of our integration by parts formula. We obtain:

$$\int x \cos x \, dx = \frac{1}{2}x^2 \cos x + \int x \sin x \, dx$$

We can see that this set of choices has left us with an integral on the right-hand side which is no simpler than the one we wanted to evaluate in the first place. Since this integral is of a similar shape to the one we want to evaluate, we might try applying the integration by parts technique once more to try and solve it. However, even if this were to be successful, it would mean one more application of the process which, using our original set of choices, solved the problem in one single step.

The obvious conclusion to draw from this phenomenon is the following: not every choice for the variables u and v is equally useful. I will revisit this issue, and offer some heuristics for making the best *choice of parts*, as we call the allocation of u and v , in a later section. First, however, in order to become thoroughly familiar with integration by parts, and in order to make some interesting observations about its use, we need to consider a number of further examples of the method in action.

4 Some Further Examples

I will now set out several examples which will illustrate the method of integration by parts in action. As in [Part 1](#), these will also allow us to proceed to make some generalizations and create a set of rules for using the method.

Example 1

I propose to start with an integral which is in some ways similar to the one with which we started this part of the book:

$$I = \int x e^x dx$$

Instead of being the product of x and a trigonometric function, this integrand has as its second factor an exponential function. Fortunately, it's one which is very easy to differentiate and integrate. So we might feel confident enough to allocate parts in a similar way to the allocation we used our opening example, as follows:

Integration by Parts

$$u = x \text{ and } \frac{dv}{dx} = e^x$$

This choice gives us the following consequences:

$$\frac{du}{dx} = 1 \text{ and } v = e^x$$

All this looks fairly simple and promising. We can therefore plug these identities into the integration by parts formula, so that our integral now turns into:

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + C$$

We might like to take the final step of factorising the result, so that we obtain:

$$I = e^x(x - 1) + C$$

which is a little neater.

This integral was very easily evaluated using integration by parts. Let's now proceed boldly on towards some more challenging integrals.

Example 2

However, one step at a time: here we will consider an integral which is only a little more complicated than our opening example. It is as follows:

$$I = \int (3x + 5) \cos \frac{1}{4}x dx$$

Here, instead of a simple first factor of x , we have instead a linear expression in x ; the second factor in our product of functions is of the form $\cos kx$, which we can differentiate using the Chain Rule, and integrate easily by substitution, or by using one of the standard results listed in [Appendix 4 Section 2](#).

By analogy with the choices we made in our opening example, then, we will start by choosing parts as follows:

$$u = 3x + 5 \text{ and } \frac{dv}{dx} = \cos \frac{1}{4}x$$

This choice gives us the following consequences:

$$\frac{du}{dx} = 3 \text{ and } v = 4 \sin \frac{1}{4}x$$

Let's now insert all these facts into the integration by parts formula. Our integral becomes:

$$\int (3x + 5) \cos \frac{1}{4}x \, dx = 4(3x + 5) \sin \frac{1}{4}x - \int 12 \sin \frac{1}{4}x \, dx$$

The right-hand side expression, after integration of the second term, produces the solution:

$$I = 4(3x + 5) \sin \frac{1}{4}x + 48 \cos \frac{1}{4}x + C$$

A little factorisation produces the tidied-up version, which is as follows:

$$I = 4 \left((3x + 5) \sin \frac{1}{4}x + 12 \cos \frac{1}{4}x \right) + C$$

We are already beginning to see from the examples considered so far that integration by parts will often integrate a product of two functions to give a solution which is the sum of more than one term. One clue as to why this might happen is to be found in the integration by parts formula itself. Its right-hand side contains two terms separated by a minus sign, so that we can be fairly confident that our solutions will contain two separate terms, at least before factorising or otherwise re-arranging. We can expect to see more behaviour of this type as we proceed.

Example 3

So far, all our examples have featured integrands which are the products of functions of two different types. On this occasion, however, we will look at an example where we have to integrate a product of two polynomials in x (one of them, admittedly, much simpler than the other!). The integral we will consider is the following:

$$I = \int x(3x + 5)^2 dx$$

Of course, we could just expand the squared term to a quadratic in x , and then multiply through by the x on the left; this would give us a polynomial in x , which we could then integrate very simply term by term. However, I'd like instead to take a look at the application of integration by parts to this kind of problem, so we will turn our backs on this temptation.

We might also remark that the integrand is of the form:

$$(ax + b)(cx + d)^p$$

(where in the present case $a = 1$ and $b = 0$) so that we might expect integration by substitution, using $u = 3x + 5$, to bring us a fairly easy solution. (Indeed, I give this solution for reference in [Appendix 5 Section 7.](#))

To reiterate, we are concerned here with integration by parts, and I have included this example to illustrate the full range of application of the technique. So we will proceed to choose parts, and watch with interest as things unfold.

Integration by Parts

If we try to allocate parts on a simple left-to-right basis, we immediately find ourselves faced with the following:

$$u = x \text{ and } \frac{dv}{dx} = (3x + 5)^2$$

We could find v using integration by substitution, but if we're going to have to do that, then we might as well just integrate the whole product of functions using substitution. So let's try reversing the choice of parts, so that we have:

$$u = (3x + 5)^2 \text{ and } \frac{dv}{dx} = x$$

This gives us these consequences:

$$\frac{du}{dx} = 6(3x + 5) \text{ and } v = \frac{1}{2}x^2$$

(The first of these results was obtained by differentiating our expression for u using the Chain Rule.)

We now insert all these pieces of information into the integration by parts formula, so that our integral becomes:

$$\int x(3x + 5)^2 dx = \frac{1}{2}x^2(3x + 5)^2 - \int 3x^2(3x + 5)dx$$

The second term on the right-hand side can be multiplied out (I think we can probably allow ourselves this luxury, having actually applied our technique), and behaves as follows:

$$\int 3x^2(3x + 5)dx = \int (9x^3 + 15x^2)dx = \frac{9}{4}x^4 + 5x^3 + C$$

The first term on the right-hand side of the integration by parts formula can be expanded into a quadratic and then multiplied out, as follows:

$$\frac{1}{2}x^2(9x^2 + 30x + 25) = \frac{9}{2}x^4 + 15x^3 + \frac{25}{2}x^2$$

Subtracting the expansion of the right-hand term from that of the left-hand one, and gathering together similar powers of x , we are left with:

$$\left(\frac{9}{2} - \frac{9}{4}\right)x^4 + (15 - 5)x^3 + \frac{25}{2}x^2 + C = \frac{9}{4}x^4 + 10x^3 + \frac{25}{2}x^2 + C$$

Probably the best final polish we can put on this expression would be to factorise in order to remove the multiple fractions, which leaves us with the following:

$$I = \frac{1}{4}x^2(9x^2 + 40x + 50) + C$$

which is, of course, a polynomial in x , and hardly different in quality from the result we might have obtained by just expanding our the integrand in the first place.

The lesson we draw from this experience – a rather more laborious one than we might have hoped for – is perhaps that integration by parts can be a promising approach when we want to integrate the product of two functions of *different* types, but it is perhaps less of a help when we have to deal with products of functions of the *same* type – at least, when the two functions in question are polynomials in x .

Example 4

With the salutary experience of [Example 3](#) under our belts, let's return to look at another situation where we are asked to integrate a product of two functions of different types. This time, however, we will explore a case involving a function of a type we have not looked at before in this context.

Our integral will be as follows:

$$I = \int x \tan^{-1} x \, dx$$

which has as its second factor an *inverse* trigonometric function.

We might at first try choosing parts in a similar order to that used in the first two examples of this section. This would give us the following choices:

$$u = x \text{ and } \frac{dv}{dx} = \tan^{-1} x$$

In order to proceed to the next stage of our usual train of reasoning, we would now need to know what expression corresponds to v . In order to be able to write this expression down would need to know the integral of $\tan^{-1} x$. Do we know what this is? It's not usually presented as one of the more common standard integrals, and I don't propose to assume knowledge of it here. (In fact, it is possible to evaluate it by parts, although with the use of a clever trick which we have not yet mentioned. We will be examining this technique in a later section.)

In order not to stumble any further in this direction, where we have already encountered a difficulty, let's consider what happens when we choose parts differently, as follows:

$$u = \tan^{-1} x \text{ and } \frac{dv}{dx} = x$$

We can immediately remark that we do in fact know how to *differentiate* $\tan^{-1} x$ - indeed, [Appendix 4 Section 2](#) gives us a standard integral which integrates to $\tan^{-1} x$; this means that its integrand must be the derivative of $\tan^{-1} x$, by the Fundamental Theorem of Calculus.

We are therefore able to proceed optimistically to draw the following conclusions from our new choice of parts:

Integration by Parts

$$\frac{du}{dx} = \frac{1}{x^2 + 1} \text{ and } v = \frac{1}{2}x^2$$

If we now substitute these identities into the integration by parts formula, our integral will become:

$$\int x \tan^{-1} x \, dx = \frac{1}{2}x^2 \tan^{-1} x - \int \frac{x^2}{2(x^2 + 1)} dx$$

At least the second term on the right-hand side no longer has any inverse trigonometric functions in it. In itself, however, it still looks a little challenging, so we will need some careful thought in order to evaluate it .

If we start by rewriting it with a constant factor taken outside the integral sign, as follows:

$$\frac{1}{2} \int \frac{x^2}{x^2 + 1} dx$$

then we might notice, after a look back at [Part 1 Section 10 Example 2](#) to refresh our memory, that an approach based on tangent-substitution might be fruitful. However, without hauling you down a frustrating blind alley, I will simply point out that the presence of the x^2 in the numerator will leave us with an unresolved $\tan^2 u$ to integrate once all the cancelling has been done. We will therefore need an alternative.

Some simple algebra on the integrand will be of service here. In [Part 1 Section 8 Example 5](#), we dealt with an awkward-looking function expressed as a fraction by splitting it into two separate fractions, which were then easily dealt with on an individual basis. In the present case, our numerator has only one term, so we will need to transform it a little first. It is always possible to leave an expression unchanged in value by adding and subtracting a constant, so we might try adding and subtracting 1 to the numerator, having sought inspiration for our choice of constant from the nature of the denominator. This will alter our integrand as follows:

$$\frac{x^2 + 1 - 1}{x^2 + 1} = \frac{x^2 + 1}{x^2 + 1} - \frac{1}{x^2 + 1} = 1 - \frac{1}{x^2 + 1}$$

Remembering our list of standard integrals, this immediately begins to look a lot easier. Indeed, the second term of our integration by parts formula for the original integral now looks like this:

$$\frac{1}{2} \int \frac{x^2}{x^2 + 1} dx = \frac{1}{2} \int \left(1 - \frac{1}{x^2 + 1} \right) dx = \frac{1}{2} (x - \tan^{-1} x) + C$$

And finally, the solution to our original integral now takes the following form:

$$\frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}(x - \tan^{-1} x) + C = \frac{1}{2}(x^2 \tan^{-1} x + \tan^{-1} x - x) + C$$

In the final right-hand side, I have made a minor rearrangement so as to gather the terms involving $\tan^{-1} x$ together.

Notes

In this last example, we had to do two things which might not be immediately obvious. First, we had to choose parts in an order different from that which worked well for our first two examples; and second, it became necessary to use some algebraic manipulation (one might almost say “an algebraic trick”) to turn the remaining integrand on the right-hand side of the integration by parts formula for our problem into two separate components, each of which was very easy to integrate.

This last technique, where we converted a rational expression in x into an expression of the form:

$$1 \pm \frac{1}{(\text{some expression in } x)}$$

is an example of a general method which can be useful in dealing with some of the simpler rational expressions which we may be asked to integrate. I explain the method in detail in [Appendix 3 Section 2](#).

Furthermore, the integral:

$$\int \frac{x^2}{x^2 + 1} dx$$

can be solved using the method of tangent-substitution which was explored in [Part 1 Section 10](#). I give a solution by this method in [Appendix 5 Section 8](#) for reference and comparison.

Example 5

To round off this section, we will now look at another integral which has some broad similarity with the one in [Example 4](#). This integral is as follows:

$$I = \int x \ln x \, dx$$

As in the previous example, we might first consider the allocation of parts as follows:

$$u = x \text{ and } \frac{dv}{dx} = \ln x$$

We can immediately see that we are now faced with a similar stumbling-block to the one we nearly fell over in [Example 4](#) above: in order to establish the value of v , we need to know what is the integral of $\ln x$. Once again, this is not usually considered to be one of the more elementary standard integrals, and once again, I will not assume it here. (You will perhaps not be surprised to learn that it can be integrated by parts, with the aid of an algebraic trick. I give the solution in a later section, and point out in addition that even if we did know what the required integral was, we would find that it actually made our original problem *more*

Integration by Parts

difficult, and not less, due to the fact that it is a longer and more complicated expression than the $\ln x$ with which we started!)

After this lengthy explanatory digression, let's revert straight away to the alternative approach we adopted in [Example 4](#) above, and set parts as follows:

$$u = \ln x \text{ and } \frac{dv}{dx} = x$$

This choice of parts gives us the following consequences (using the standard derivative of $\ln x$):

$$\frac{du}{dx} = \frac{1}{x} \text{ and } v = \frac{1}{2}x^2$$

The integration by parts formula now transforms our integral as follows:

$$\int x \ln x \, dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x \, dx$$

where the integrand on the right-hand side is obtained after simple cancellation of the x in the denominator of the derivative of u with the x^2 which appears in our expression for v .

Evaluation of the right-hand side integral then produces the following final solution:

$$\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C = \frac{1}{4}x^2(2 \ln x - 1) + C$$

Notes and Conclusions

In all the above examples, we have been considering simple products of functions. Indeed, these have been somewhat artificially chosen, in that they are all (apart from [Example 2](#)) of the form $xf(x)$, where the place of $f(x)$ has been taken successively by functions of several different kinds.

We have found that the most useful choice of parts is different for the different types of function occupying the place of $f(x)$. If we also take into account the example with which we started this part of the book, then we can summarise the choices of parts we made in the preceding examples in the following table:

Integrand: $xf(x)$	Choice of u	Choice of dv/dx
xe^x	x	e^x
$x \cos x$	x	$\cos x$
$x \tan^{-1} x$	$\tan^{-1} x$	x
$x \ln x$	$\ln x$	x

From this information, we can see that where $f(x)$ was an exponential or trigonometric function, it was satisfactory to choose:

Integration by Parts

$$u = x \text{ and } \frac{dv}{dx} = f(x)$$

Leaving out of consideration [Example 3](#) (which, as we have already noted was ideal for the method of substitution, and only included here to provide a comparison between our two methods), we can also note that where $f(x)$ was an *inverse* trigonometric function or a logarithmic function, then we were able to proceed more simply by making the following choice:

$$u = f(x) \text{ and } \frac{dv}{dx} = x$$

Can we draw any conclusions as to why this has turned out to be the case? Well, realising that in order to use the method of integration by parts we have to split our integrand into two factors: u and dv/dx , a look back at the integration by parts formula will confirm that in order to produce the expression given by the right-hand side of that formula, dv/dx has to be integrated in order to produce v , and u has to be differentiated in order to produce du/dx .

We can therefore recommend that u should be chosen such that it is easy to differentiate, and dv/dx should be chosen so that it is easy to integrate. We can go further, especially in the light of [Examples 4 and 5](#) above, as well as the shape of the integral:

$$\int \frac{du}{dx} v dx$$

which appears on the right-hand side of the integration by parts formula, and say that we would ideally like du/dx to be *simpler* than u (or at least to introduce no more difficulty from the point of view of integration). Likewise, we can say that we would like v to be *simpler* than dv/dx (or, once again, no more complicated from the viewpoint of integration).

As we have seen above, it is not essential (or even likely) that *both* of these aspirations to simplicity will be met; but it is highly desirable that *one* of the two should be met – otherwise, the integral on the right-hand side of our formula will be no simpler than the one we started out with.

In [Examples 4 and 5](#) above, we chose u to be first $\tan^{-1} x$ and then in the next example $\ln x$. In both of these cases, du/dx produced a rational function in x , which together with the other element in our choice of parts:

$$\frac{dv}{dx} = x, \text{ so that } v = \frac{1}{2}x^2$$

left us with an integral on the right-hand side which was easier to handle than one involving inverse trigonometric or logarithmic functions.

This is probably about as far as we can go in drawing general conclusions about how best to choose parts at present. I will return to these issues, which have a bearing on the kinds of integral which we can integrate by parts as well as the statement of an algorithm for the method, a little further on.

5 Repeated Integration by Parts

All of the relatively simple integrands we examined in the previous section could be dealt with by one simple application of the integration by parts technique. There are, however, many integrands where integrating by parts produces an integral on the right-hand side which is only a little less difficult than the one we started with. In such cases, it is sometimes possible to apply integration by parts for a *second* time (or even, as we shall see, *several* more times), in order to deal with the slightly less difficult integral on the right-hand side.

We will now look at some examples of this approach at work.

Example 1

First of all, we will move one simple step beyond the integrands we considered in the previous section and try to integrate the following example:

$$I = \int x^2 \sin x \, dx$$

where we have a trigonometric function multiplied by an factor of x^2 , rather than the simple factor of x which was present in the trigonometric function example above.

Stepping in boldly in the light of the above discussion on how our various choices of parts in [Section 4](#) worked out, let us try the following:

$$u = x^2 \text{ and } \frac{dv}{dx} = \sin x$$

These choices give rise to the following results:

$$\frac{du}{dx} = 2x \text{ and } v = -\cos x$$

Putting these into the integration by parts formula, we obtain the following expression for our integral:

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2 \int x \cos x \, dx$$

(I have taken the coefficient of 2 which is part of our expression for du/dx outside the integral sign on the right-hand side so that we can see more clearly the nature of our remaining integral. Additionally - no pun intended - the minus sign in the integration by parts formula has been transformed into a plus sign by the negative sign in our expression for v , which we will recall turned out to be $-\cos x$.)

You will note that, while we cannot evaluate the integral on the right-hand side using our simple rules for integration, the integral is identical to the one we evaluated in [Section 3](#) above, as our very first illustration of the use of integration by parts. There, we used integration by parts to find the following solution:

Integration by Parts

$$\int x \cos x \, dx = x \sin x + \cos x + C$$

We can therefore substitute this solution for the right-hand side integral in order to find a final expression for I , which can be written as follows (not forgetting that the right-hand side integral must be multiplied by a coefficient of 2):

$$I = 2\cos x + 2x \sin x - x^2 \cos x + C$$

This can be thought of as a quadratic in x with trigonometric functions as its coefficients. I have rearranged it in ascending order of powers of x in order to display it without a leading minus sign.

If we feel more comfortable factorising out the two instances of $\cos x$, we can end up with a linear expression in $\sin x$ and $\cos x$, as follows:

$$I = 2x \sin x + (2 - x^2) \cos x + C$$

Note

Although we simply quoted a solution we had obtained earlier in order to complete this example quickly, it should not be lost on you that in order to evaluate our chosen integral, we did have to apply integration by parts *twice*, since after the first application, the remaining integral on the right-hand side was still not simple enough to be dealt with using direct methods.

Example 2

We now proceed to another example which will require multiple applications of our technique. This time, the integral under consideration will be:

$$I = \int x^3 e^{2x} \, dx$$

Noting the broad similarity in the nature of the two functions whose product forms our integrand to those we encountered in [Section 4 Example 1](#), we will start by choosing parts as follows:

$$u = x^3 \text{ and } \frac{dv}{dx} = e^{2x}$$

If we make use of the standard integral for e^{kx} which is quoted in [Appendix 4 Section 2](#), we can then write down the following consequences of these choices:

$$\frac{du}{dx} = 3x^2 \text{ and } v = \frac{1}{2} e^{2x}$$

The integration by parts formula now gives us:

Integration by Parts

$$I = \int x^3 e^{2x} dx = \frac{1}{2} x^3 e^{2x} - \int \frac{3}{2} x^2 e^{2x} dx$$

Let's now use the symbol I_1 to represent the integral on the right-hand side, as follows:

$$I_1 = \int \frac{3}{2} x^2 e^{2x} dx$$

We will now proceed to apply integration by parts once more, in order to simplify this integral. In line with our previous selection, we choose parts as follows:

$$u = \frac{3}{2} x^2 \text{ and } \frac{dv}{dx} = e^{2x}$$

We can deduce the following consequences:

$$\frac{du}{dx} = 3x \text{ and } v = \frac{1}{2} e^{2x}$$

The integration by parts formula then gives us this result for I_1 :

$$I_1 = \int \frac{3}{2} x^2 e^{2x} dx = \frac{3}{4} x^2 e^{2x} - \int \frac{3}{2} x e^{2x} dx$$

Noting how the integrals on the right-hand side of the integration by parts formula are becoming steadily less complicated (because of the successive reduction in the power of x each contains), we now set I_2 to represent the integral on the right-hand side of our expression for I_1 , as follows:

$$I_2 = \int \frac{3}{2} x e^{2x} dx$$

One more application of integration by parts should see us almost home and dry. Sticking with the convention which has brought us this far, we choose parts, using the same reasoning as follows:

$$u = \frac{3}{2} x \text{ and } \frac{dv}{dx} = e^{2x}$$

Just like before, we can deduce the following consequences:

$$\frac{du}{dx} = \frac{3}{2} \text{ and } v = \frac{1}{2} e^{2x}$$

The integration by parts formula then gives us this result for I_2 :

$$I_2 = \int \frac{3}{2} x e^{2x} dx = \frac{3}{4} x e^{2x} - \int \frac{3}{4} e^{2x} dx$$

Integration by Parts

In line the practice we have established, we can set I_3 to represent the integral on the right-hand side of this most recent equation, as follows:

$$I_3 = \int \frac{3}{4} e^{2x} dx$$

We will find that we can integrate this directly, to obtain the following solution:

$$I_3 = \frac{3}{8} e^{2x} + C$$

We now have no more difficult integrals to deal with using integration by parts, so that we can start to unravel the effect of the repeated uses of the technique in order to obtain an expression first for I_2 , then for I_1 , and finally for I itself. We start by substituting the solution for I_3 into the equation for I_2 . We obtain the following:

$$I_2 = \frac{3}{4} x e^{2x} - \frac{3}{8} e^{2x} + C$$

This can now be plugged into our equation for I_1 , from which we obtain:

$$I_1 = \frac{3}{4} x^2 e^{2x} - \frac{3}{4} x e^{2x} + \frac{3}{8} e^{2x} + C$$

Finally, this can be fed into the equation for I which resulted from our very first application of integration by parts. So that we end up with this solution:

$$I = \frac{1}{2} x^3 e^{2x} - \frac{3}{4} x^2 e^{2x} + \frac{3}{4} x e^{2x} - \frac{3}{8} e^{2x} + C$$

Enthusiasts of factorisation and the tidying up of repeated fractional coefficients may prefer the following rearrangement:

$$I = \frac{1}{8} e^{2x} (4x^3 - 6x^2 + 6x - 3) + C$$

It will be instructive to look at an expression for the original integral I in terms of the various intermediate integrals to which we gave labels of the form I_k . Once more, we can work back from the expression for I_2 in terms of I_3 (the first integral produced by our technique which we could solve directly) to display an expression for I in terms of all the intermediate integrals. We will obtain the following sequence of equations:

$$I_2 = \frac{3}{4} x e^{2x} - I_3$$

$$I_1 = \frac{3}{4} x^2 e^{2x} - I_2 = \frac{3}{4} x^2 e^{2x} - \frac{3}{4} x e^{2x} + I_3$$

$$I = \frac{1}{2} x^3 e^{2x} - I_1 = \frac{1}{2} x^3 e^{2x} - \frac{3}{4} x^2 e^{2x} + I_2 = \frac{1}{2} x^3 e^{2x} - \frac{3}{4} x^2 e^{2x} + \frac{3}{4} x e^{2x} - I_3$$

In the equations for I and I_1 , we see that the minus sign before I_1 and I_2 respectively causes the signs in the expansions of these integrals to be reversed. In particular, the sign preceding I_3 is made to alternate from minus to plus and then back to minus again as we expand our solution, moving from the equation for I_2 back up to the equation for I .

Setting out our working in this way helps to see how the final solution, comprising several terms separated by addition or subtraction signs, is built up by repeated applications of the integration by parts technique. We may gain an even more powerful overview of this process, however, by setting it out in the form of a table, as shown below:

Stage	$\int u \frac{dv}{dx} dx$	u	$\frac{dv}{dx}$	uv	$\int \frac{du}{dx} v dx$
1 (I)	$\int x^3 e^{2x} dx$	x^3	e^{2x}	$\frac{1}{2} x^3 e^{2x}$	$\int \frac{3}{2} x^2 e^{2x} dx$
2 (I_1)	$\int \frac{3}{2} x^2 e^{2x} dx$	$\frac{3}{2} x^2$	e^{2x}	$\frac{3}{4} x^2 e^{2x}$	$\int \frac{3}{2} x e^{2x} dx$
3 (I_2)	$\int \frac{3}{2} x e^{2x} dx$	$\frac{3}{2} x$	e^{2x}	$\frac{3}{4} x e^{2x}$	$\int \frac{3}{4} e^{2x} dx$
4 (I_3)	$\int \frac{3}{4} e^{2x} dx$			Solution $\frac{3}{8} e^{2x}$	

This table contains one line for each application of integration by parts; each stage is numbered, and the integral to be solved at each stage is shown in column 2. Columns 3 and 4 show the choice of parts made at each stage, while the final two columns – 5 and 6 – show the two components of the right-hand side of the integration by parts formula for each application. Row 4 is unique, since the integral in column 2 could be solved directly, without a further integration by parts; I have, nonetheless, written its solution in the column headed “ uv ”, to help us to read off the complete solution from the table.

The final solution to the integral we started out with can be obtained, then, by reading off each entry in the “ uv ” column, starting at the top and proceeding down to the bottom row. Entries must, however, be separated by the appropriate sign. What is this “appropriate sign”? Well, we must clearly start with a minus sign, since the integration by parts formula dictates it. Subsequently, however, we must *alternate signs*, because the minus sign in the integration by parts formula for one stage will change the minus sign obtained at the next stage to a plus sign, by the following standard rule of algebra:

$$a - (b - c) = a - b + c$$

At the succeeding stage, however, our integration by parts formula will be preceded by a plus sign, so that its own native minus sign will remain when we expand out our result. This process can be seen at work in the description of the process I gave a few paragraphs earlier in terms of the various intermediate integrals with names of the form I_k .

Reading our solution off the “ uv ” column, and using alternating signs as suggested by the above reasoning, we will obtain:

Integration by Parts

$$I = \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}e^{2x} + C$$

remembering that we must be sure to add the constant of integration C . As we would expect, this is indeed the solution we obtained by our original stage-by-stage process.

Example 3

The previous two examples have entailed quite a lot of explanation, which has arisen in part from the very fact that integration by parts has had to be applied repeatedly. In [Example 2](#) above, however, we ended the discussion by displaying a table which summarised the working we were obliged to undertake to find our solution. In this example, I propose to dispense in large part with the kind of detailed explanation I have presented so far, and instead evaluate the chosen integral using a table-based approach alone.

The integral which will be the subject of this endeavour will be the following (not completely dissimilar to the previous two examples):

$$I = \int x^3 \cos x \, dx$$

Here is the table which sets out the procedure for evaluating this indefinite integral:

Stage	$\int u \frac{dv}{dx} dx$	u	$\frac{dv}{dx}$	uv	$\int \frac{du}{dx} v dx$
1	$\int x^3 \cos x \, dx$	x^3	$\cos x$	$x^3 \sin x$	$\int 3x^2 \sin x \, dx$
2	$\int 3x^2 \sin x \, dx$	$3x^2$	$\sin x$	$-3x^2 \cos x$	$-\int 6x \cos x \, dx$
3	$-\int 6x \cos x \, dx$	$6x$	$-\cos x$	$-6x \sin x$	$-\int 6 \sin x \, dx$
4	$-\int 6 \sin x \, dx$			Solution $6 \cos x$	

Let's record a few remarks about this table, before we read off from it the solution we require.

1. The integral in row 4 and column 2 can be solved directly, without further use of integration by parts. Just as we did in [Example 2](#), then, we write down its solution in the " uv " column, to help us read off the full solution easily.
2. In completing this table, we work through each row in turn, making our way from left to right. Column 2 always contains the integral we have to deal with at each particular stage of the repeated process. From this column, it is generally quite easy to complete columns 3 and 4, which show our choice of parts for the next repetition of integration by parts.
3. Columns 4 and 5 are the only ones which really require any work from us: in order to complete column 4, we must deduce v from the expression for dv/dx shown in column 3 (in other words, we must perform a quick bit of *integration*). For column 5,

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we will need to obtain du/dx from the expression for u which is found in column 3 (that is to say, we must *differentiate*).

4. We can be sure that the process has finished when we find that column 2 contains an integral which can be evaluated directly.

Having made these comments, which are intended to be helpful in using this tabular approach to deal with other integrals, we can now reap the benefits of this shorthand method and simply read off our solution from the “ uv ” column, as we did at the conclusion of [Example 2](#).

We read off the expressions in the “ uv ” column, separating each expression with a sign alternating between minus and plus, as discussed above. We start with the minus sign which is dictated by the integration by parts formula, and we must be sure to take note of any signs which precede the various expressions for uv , since these may interact with our “default” sign to change what we actually write down.

Proceeding in this way, we will arrive at the following solution:

$$I = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + C$$

The signs between the terms should have been (in order) *minus, plus, minus*. However, minus signs at the start of the expressions for uv in rows 2 and 3 reversed the first two of these, so bringing us to the above result. We have, of course, added a constant of integration labelled C at the very end.

This tabular approach is clearly much easier to set out and allows the process of repeated integration by parts to be made fairly straightforward, depending as it does on a simple application of integration and one of differentiation in completing each row, as well as the simple production of the result by reading off the entries in the “ uv ” column. However, it can be made even more straightforward and “mechanical” by a refinement of the table layout, which I will explain in the next section.

6 Tabular Integration by Parts

In this section, I will set out a refined version of the tabular approach to repeated integration by parts which I started to describe in the previous section. I will start by illustrating the refined table layout which I propose with [Example 3 from, the previous section](#), which will allow the new table layout to be compared with the one we saw at the end of [Section 5](#).

By way of reminder, the integral we want to evaluate in [Section 5 Example 3](#) was:

$$I = \int x^3 \cos x \, dx$$

And here is a simplified table which sets out the most important elements from the more complex table I used above:

u	$v (dv/dx = \cos x)$	<i>Sign</i>
x^3	$\sin x$	–
$3x^2$	$-\cos x$	+

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6x	− sin x	−
6	cos x	+
0		

This table is intended to be simpler and easier to work through than the more comprehensive table I began to set out towards the end of the previous section. Here, we only have three simple columns – one listing values of u , one listing values of v , and a third column corresponding exactly to the convention on the sign linking each term of the result which we used in [Section 5](#) above.

So how do we use this new tabular display of our repeated integration by parts operations? Some simple instructions are as given below. Before we start, however, we can make one observation about how we set up the table in the first place. Bearing in mind that integration by parts requires us to *choose parts*, that is, to express the integral we want to evaluate in the form:

$$\int u \frac{dv}{dx} dx$$

and since this new style of table only has columns corresponding to u and v , we can start by putting the part of our integrand which we have chosen to be dv/dx in the heading for the v column. Having done that, all of the subsequent entries in this column will correspond to a value of v at some stage in our repeated process.

Having made this remark, let's proceed to our instructions:

1. In the first row, enter the value of u which results from our initial choice of parts, and the value of v which we obtain from integrating the dv/dx part of our integrand. This part has already been entered in the column heading.
2. We can obtain entries for each subsequent row by *differentiating* the u entry in the previous row, and *integrating* the v entry in the previous row. (For convenience, we omit the constant of integration.)
3. Repeat this process until the u column is found to contain a value of 0. This means that the process is at an end.
4. Fill in the *Sign* column with entries alternating between *plus* and *minus*, starting in the first row with a *minus* sign.

Having completed our table, it is then a very simple matter to read off the solution, multiplying together the values for u and v from each row in turn, and following this uv value with the sign given in the third column. We must, of course, keep in mind the possibility that the sign in the third column may well combine with a negative sign attached to the next uv instance to produce an altered sign in the final solution. The last line of all, which contains a zero in the u column and marks the termination of the tabular process, contributes nothing to our solution. Naturally, we must remember to add a constant of integration C to our final solution in order for it to be fully valid.

Proceeding to read off our solution in this way, we will, of course, obtain the same solution as was extracted from our earlier, more complex table in [Section 5](#), namely:

$$I = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + C$$

In order to provide some more examples of the tabular approach I have set out here, I will now give brief illustrations of the tabular solutions to [Examples 1 and 2 from Section 5](#) above.

Example 1

In this example, our integral is:

$$I = \int x^2 \sin x \, dx$$

and the table which gives its solution is as follows:

<i>u</i>	<i>v</i> (<i>dv/dx</i> = sin <i>x</i>)	<i>Sign</i>
x^2	$-\cos x$	$-$
$2x$	$-\sin x$	$+$
2	$\cos x$	$-$
0		

We can now read off our solution as follows:

$$I = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

and this can be rearranged in the way we used in [Section 5](#) to avoid an opening minus sign as follows:

$$I = 2 \cos x + 2x \sin x - x^2 \cos x + C$$

Example 2

Here, we want to integrate:

$$I = \int x^3 e^{2x} \, dx$$

and from it, we obtain the following table for our repeated integration by parts:

<i>u</i>	<i>v</i> (<i>dv/dx</i> = e^{2x})	<i>Sign</i>
x^3	$\frac{1}{2}e^{2x}$	$-$
$3x^2$	$\frac{1}{4}e^{2x}$	$+$
$6x$	$\frac{1}{8}e^{2x}$	$-$
6	$\frac{1}{16}e^{2x}$	$+$
0		

Integration by Parts

Reading off our solution, row by row and left to right, making sure we apply the sign in the third column at each stage, we obtain the following solution:

$$I = \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}e^{2x} + C$$

(We might pause here to note how much easier the tabular method makes this example, where we are beset on all sides by hard-to-track fractional coefficients.)

Example 3

As a final example of the use of the tabular approach to integration by parts, we will demonstrate that the method does, of course, work perfectly well for integrals where only one application of the method is called for. As we might expect, such a table has few rows, and the process of generating the table which we have been using comes to an end very quickly. However, there may be some notational convenience in applying integration by parts using the tabular method – it is certainly very quick to write down!

Let's apply the tabular approach to the very first integral we evaluated using the method of integration by parts:

$$I = \int x \cos x \, dx$$

Our table will look like this:

<i>u</i>	<i>v (dv/dx = cos x)</i>	<i>Sign</i>
x	sin x	–
1	– cos x	+
0		

From this table (shorter than our other examples, but still with enough rows to be useful) we can read off our solution as follows:

$$I = x \sin x + \cos x + C$$

As we would by now expect, this is the very solution we obtained earlier using the “longhand” method.

Why Does this Method Work?

It will be interesting to look briefly at some general reasoning which gives us an insight into exactly why this tabular approach to integration by parts works.

Let's start with an integral I where we can divide the integrand (a product of two functions) into parts in the following way:

$$I = \int u_1 \frac{dv_1}{dx} dx$$

Integration by Parts

I have given the letters u and v subscripts of 1 to indicate that this is the first stage in a potentially multi-stage process of integration by parts. We should also note at this stage that the tabular approach comes into its own when we choose as u_1 a factor of our product of functions which we can expect to be “differentiated away” – ie reduced by repeated differentiation to a eventual value of zero.

The integration by parts formula gives us the following expression for the integral I :

$$I = u_1 v_1 - \int \frac{du_1}{dx} v_1 dx$$

Naming the integral on the right-hand side I_1 , we will select parts for its integration as follows (bearing in mind that our original integrand is one where we expect the u -part to respond to repeated differentiation):

$$u_2 = \frac{du_1}{dx} \text{ and } \frac{dv_2}{dx} = v_1$$

Once more, the integration by parts formula gives us:

$$I_1 = u_2 v_2 - \int \frac{du_2}{dx} v_2 dx$$

Substituting this back into our original formula for I will give us:

$$I = u_1 v_1 - I_1 = u_1 v_1 - u_2 v_2 + \int \frac{du_2}{dx} v_2 dx$$

Note particularly here that the sign in front of the right-hand side integral is now a plus sign, since it is the result of the application of a minus sign to the formula for I_1 .

Let's take this step-by-step approach one stage further before we start looking at what it is telling us about the overall process. We will set:

$$I_2 = \int \frac{du_2}{dx} v_2 dx$$

We will choose parts as follows:

$$u_3 = \frac{du_2}{dx} \text{ and } \frac{dv_3}{dx} = v_2$$

According to the integration by parts formula:

$$I_2 = u_3 v_3 - \int \frac{du_3}{dx} v_3 dx$$

Substituting this back into our original formula for I will give us:

Integration by Parts

$$I = u_1v_1 - u_2v_2 + I_2 = u_1v_1 - u_2v_2 + u_3v_3 - \int \frac{du_3}{dx}v_3dx$$

We can now see a pattern beginning to emerge: the expression for the integral I which we are building up is formed of a series of terms of the form u_kv_k . Each successive u_k is obtained by differentiating u_{k-1} , as is shown in the choice of parts we make above at each stage. Additionally, the choice of parts in each case also shows us that v_k is obtained at each stage from the formula:

$$\frac{dv_k}{dx} = v_{k-1} \text{ which means that } v_k = \int v_{k-1}dx$$

In other words, we obtain v_k by *integrating* v_{k-1} , since integration can be regarded as the inverse process to differentiation. Finally, the sign connecting each u_kv_k term is in the first instance a minus sign (since this is what the integration by parts formula produces); next time, it is a plus sign, due to the application of a minus sign to the minus sign which comes out of a second application of the formula; and because of this plus sign, the next sign to come out of the process will be an unvarnished minus sign. And so we could go on, cycling between minus and plus every time we repeat the integration by parts process.

Now what I have described here is exactly what we are instructed to do when we complete each line of the table in the method we have been looking at in this section – once we have written down the values of u and v in the first line of the table, we proceed to each subsequent line by differentiating the function in the u column and integrating the function in the v column.

What happens, then, when we reach the end of the process? In the examples we have looked at here, we eventually reached an instance of u_k - let us call it u_n - where one more differentiation step will produce a result of zero. In symbols, we will have the following situation:

$$\frac{du_n}{dx} = 0$$

If this is so, then we will find that we have an integral I_{n-1} in our chain of integrals which we want to evaluate by parts, and the integration by parts formula will give us the following:

$$I_{n-1} = u_nv_n - \int \frac{du_n}{dx}v_ndx$$

Because, as explained above, we are looking here at a situation where the derivative of u_n is zero, the integral on the right-hand side of this equation is itself zero, and we find that we have no more integration operations to perform. Hence we can state that:

$$I_{n-1} = u_nv_n$$

and the process of repeated integration has come to an end, since we have no more integrals to evaluate. We have described here the final stage in the creation of the table.

Finally, then, we must write down the result of the overall integration task. We have done this in the examples in this section by multiplying together the u and v entries from each line in our table, and combining them using the alternating plus and minus signs we set out in the third column. This results in an expression somewhat like the final expression we wrote down earlier for the integral I in terms of the first three entries for u and v . We can write down the general result of the tabular process as follows:

$$I = u_1v_1 - u_2v_2 + u_3v_3 - \dots + (-1)^{n-1}u_nv_n + C$$

The expression $(-1)^{n-1}$ is a shorthand method of specifying the way the plus and minus signs vary between the terms of the right-hand side expression. If n is even, then $n - 1$ is odd, and an odd power of -1 is itself -1 – so we will have a minus sign in front of every u_kv_k where k is even. Similarly, if n is odd, then $n - 1$ is even, and an even power of -1 results in 1, without the minus sign. This gives us the result we need for the sign preceding u_kv_k where k is odd.

We have, of course, added the usual constant of integration C at the end of our expression.

As a concluding note, we can write the expression for I using sigma-notation, once again taking advantage of the shorthand method we just used to force the signs to come out correctly. This will appear as follows:

$$I = \sum_{i=1}^n (-1)^{i-1} u_i v_i + C$$

The reasoning we have given here will provide some justification for our practice when completing the third column of our table, which is to enter a minus sign in the first row, and then proceed downwards, alternating between a minus sign and a plus sign.

7 How to Choose Parts

Having drawn some conclusions at the end of [Section 4](#) above based on our first set of examples, and having also built up a small fund of evidence about repeated integration by parts in [Sections 5 and 6](#), we are now in a position to give a slightly more useful rule of thumb for choosing parts.

In my note at the end of [Section 4](#), I remarked that when choosing parts, it can be useful to choose u to be the part of our integrand which is easier to differentiate, and to let dv/dx be the part which is easier to integrate, if such choices can easily be made. In the examples in [Sections 5 and 6](#), we found that if we consistently choose parts according to these preferences, then in our repeated applications of integration by parts will cause the part chosen as u to “differentiate away” after several applications of the technique, leaving us with an integral at the end which can often be solved directly.

We should note at this point that, particularly for functions which are *algebraic* functions (ie they are composed of a sum of terms of the form kx^n) then repeated applications of differentiation will gradually reduce the index of each term until we are finally left with 0. We should remind ourselves here that the rules for tabular integration by parts specifically state that the process ends when the u column has “differentiated away” to zero!

We can summarise the experience we have gained with a simple mnemonic: **LIATE**. Here, each letter stands for a particular type of function, as follows:

Letter	Meaning	Examples
L	Logarithmic	$\ln x, \log_a x$
I	Inverse trigonometric	$\sin^{-1} x, \tan^{-1} x$
A	Algebraic	$x^2, 6x^4$
T	Trigonometric	$\cos x, \tan x$
E	Exponential	$e^x, 2^x$

When choosing parts, we should choose u to be the function in our integrand which comes first in this list of function types, and set dv/dx to be the other part of the integrand. We can compare this table with its associated mnemonic with the table at the end of [Section 4](#), which provided a summary of our experience of the best choice of parts for the examples in that section. We will see that every line in that table agrees with the choice which we would make based on the mnemonic I have just given.

(This mnemonic was proposed by Herbert Kasube of Bradley University, and is sometimes given in the opposite order, as **DETAIL**. Here, the initial letter D can be taken, using the conventions I have adopted in this book, to stand for dv/dx , and the mnemonic becomes a rule for selecting this part of the integrand instead of u .)

We have so far always ended up choosing parts in the way this mnemonic recommends. In [Examples 4 and 5 of Section 4](#), we started out by trying to set u to be the algebraic part of the integrand, only to find that this did not lead to a very easy ride. Instead, we found that choosing u to be the inverse trigonometric or logarithmic function did make it possible to evaluate the integrals. In [Example 4](#), admittedly, there was a need for some algebra to finish things off, but we did obtain a solution in the end. In all our other examples, including those which called for repeated integration by parts, we chose u to be the algebraic part of the integrand, and dv/dx to be the exponential or trigonometric part. All of this is precisely in accord with the recommendations of the **LIATE** mnemonic.

We should, however, admit that all of our examples so far have contained integrands of the form $P(x)f(x)$, where $P(x)$ has been a polynomial in x (sometimes a very simple one), and $f(x)$ some other function of a type contained somewhere in the **LIATE** list. We can certainly conclude that functions of this type can be readily integrated by parts (sometimes requiring repeated applications), although we only have limited experience of examples where $f(x)$ is an inverse trigonometric or logarithmic function. I shall attempt to remedy this lack by ending this section with two more examples of this type, both slightly more complicated than those we have seen so far.

Example 1

Here we will examine the following integral:

$$I = \int (2x - 1) \ln(x^2 + 1) dx$$

This is certainly of the form we described in the previous paragraph, and our mnemonic for choosing parts dictates the following approach:

$$u = \ln(x^2 + 1) \text{ and } \frac{dv}{dx} = 2x - 1$$

These choices allow us to draw the following conclusions:

$$\frac{du}{dx} = \frac{2x}{x^2 + 1} \text{ and } v = x^2 - x$$

(The expression for du/dx is obtained by an application of the Chain Rule.)

We can now apply the formula for integration by parts to obtain the following expression for our original integral:

$$\int (2x - 1) \ln(x^2 + 1) dx = (x^2 - x) \ln(x^2 + 1) - \int (x^2 - x) \frac{2x}{x^2 + 1} dx$$

By multiplying out, we can rewrite the integral on the right-hand side as follows:

$$I_1 = \int \frac{2x^3 - 2x^2}{x^2 + 1} dx$$

Our work is now reduced to the task of evaluating this integral, after which we will be able to write out a complete solution to our original problem. The integral I_1 is certainly no pushover, but we have at least “differentiated away” the logarithmic function in the integrand with which we began. And since this new integrand is now a rational expression, we might hope that some algebraic manipulation will allow us to turn it into something more easily integrable.

I will start the ball rolling by performing a very simple piece of algebraic manipulation, simply splitting this rational function into two parts, as follows:

$$\frac{2x^3 - 2x^2}{x^2 + 1} = \frac{2x^3}{x^2 + 1} - \frac{2x^2}{x^2 + 1}$$

We will proceed to evaluate I_1 by integrating each of these two parts separately, and then combining the results at the end.

So we will first consider:

$$I_2 = \int \frac{2x^3}{x^2 + 1} dx$$

Now, readers who have been paying close attention during their perusal of this book will have noticed that this is just the very integral we evaluated by substitution in [Example 4 of Part 1 Section 8](#), on algebraic substitutions, multiplied by a factor of 2. (We were able to generalise from it in order to reveal a new class of integrals which could be dealt with by substitution.)

Integration by Parts

Therefore, taking the solution we obtained in the substitution example and multiplying it by 2, we obtain the following solution for I_2 :

$$I_2 = x^2 - \ln(x^2 + 1) + C$$

Now we can move on to the second of our two integrals which is:

$$I_3 = \int \frac{2x^2}{x^2 + 1} dx$$

Once again, we can reap benefits from the work we have already done, by noting that we have also encountered this integral (without its factor of 2) before. In [Section 4 Example 4](#) above, we had to evaluate I_3 without its factor of 2 in the course of using the integration by parts technique, and we did it using an algebraic rearrangement. (We also noticed at that time that the integral could be evaluated using tangent-substitution, and I have given a solution by this method in [Appendix 5 Section 8](#).)

So once again, taking the solution we obtained earlier (by either method!) and multiplying it by 2, we have the following solution for I_3 :

$$I_3 = 2(x - \tan^{-1} x) + C$$

We should now be in a position to assemble a complete solution to our original integral. As a first step, remembering how we divided I_1 into its two component parts, we can state that:

$$I_1 = I_2 - I_3 = x^2 - \ln(x^2 + 1) - 2(x - \tan^{-1} x) + C$$

We might consider it useful to multiply out the bracketed term and rearrange this expression so that like terms are put next to one another, giving the following:

$$I_1 = x^2 - 2x - \ln(x^2 + 1) + 2 \tan^{-1} x + C$$

Our final step will be to go back to the expression we obtained using the formula for integration by parts, and write out the following solution for the original integral:

$$I = (x^2 - x) \ln(x^2 + 1) - x^2 + 2x + \ln(x^2 + 1) - 2 \tan^{-1} x + C$$

A final rearrangement to gather together all the terms in $\ln(x^2 + 1)$, still keeping like terms together, will give us the following:

$$I = (x^2 - x + 1) \ln(x^2 + 1) - 2 \tan^{-1} x - x^2 + 2x + C$$

Notes

First of all, we notice with a passing interest that an inverse tangent function has had to be introduced to express the integral of the product of a polynomial and a logarithmic function.

Of particular relevance to our current topic, however, will be the remark that by choosing parts according to the **LIATE** mnemonic, we were able to remove the logarithmic function

from consideration, and by means of the integration by parts formula, provide ourselves with a rational function to integrate. While initially daunting, we found it possible to split this into two parts, both of which were very close to functions we had already learnt how to integrate.

If we had made a different choice of parts, we would immediately have had to face the prospect of integrating $\ln(x^2 + 1)$ in order to apply the integration by parts formula at all, which might have been an even more distressing prospect.

Example 2

In order to round off our look at the importance and usefulness of making the correct choice of parts, we will now consider another integral which will be made relatively approachable by the correct choice of parts.

This integral will be as follows:

$$I = \int (3x^2 - 4) \tan^{-1} x \, dx$$

As in the previous example, the **LIATE** rule leads us to the following choice of parts:

$$u = \tan^{-1} x \text{ and } \frac{dv}{dx} = 3x^2 - 4$$

This leads us to the following conclusions:

$$\frac{du}{dx} = \frac{1}{x^2 + 1} \text{ and } v = x^3 - 4x$$

That said, the integration by parts formula now gives us the following equation:

$$\int (3x^2 - 4) \tan^{-1} x \, dx = (x^3 - 4x) \tan^{-1} x - \int (x^3 - 4x) \frac{1}{x^2 + 1} \, dx$$

Again, after multiplying out, the integral on the right-hand side becomes:

$$I_1 = \int \frac{x^3 - 4x}{x^2 + 1} \, dx$$

presenting us with another rational integrand similar in appearance to the one we arrived at in the previous example.

We will again split the integrand into two parts as follows:

$$\frac{x^3 - 4x}{x^2 + 1} = \frac{x^3}{x^2 + 1} - \frac{4x}{x^2 + 1}$$

and examine the integration of each term separately.

If we first consider:

$$I_2 = \int \frac{x^3}{x^2 + 1} dx$$

we first note that this is almost the same I_2 which we had to consider in the previous example, and then go on to remark that it is in fact exactly the integral we evaluated by substitution in [Example 5 of Part 1 Section 8](#). We can therefore write down its solution without further ado as:

$$I_2 = \frac{1}{2}(x^2 - \ln(x^2 + 1)) + C$$

Let's move straight on from this simple step to our second integral, which is:

$$I_3 = \int \frac{4x}{x^2 + 1} dx$$

If we rewrite this integral in the following way:

$$2 \int \frac{2x}{x^2 + 1} dx$$

Then we will notice that the expression which remains under the integral sign is in fact a function-derivative pair, and as such is in the precise form we look for in order to apply straightforward u -substitution. Using the notation of [Part 1 Section 5](#), the expression has the form $g'(f(x))f'(x)$, where:

$$g'(f(x)) = \frac{1}{x^2 + 1}; f(x) = x^2 + 1; f'(x) = 2x$$

We further note that this allows us to say:

$$g'(x) = \frac{1}{x} \text{ so that } g(x) = \ln|x|$$

and that in [Part 1 Section 5](#) we established that the solution to this integral is:

$$g(f(x)) + C = \ln(x^2 + 1) + C$$

Restoring the factor of 2 which we earlier moved outside the integral sign in order to make the appropriateness of substitution clearer, we can now state that:

$$I_3 = 2 \ln(x^2 + 1) + C$$

By this point, we have the solution to the integral of the rational function which we obtained on the right-hand side of our integration by parts formula. This is simply:

$$I_1 = I_2 - I_3 = \frac{1}{2}(x^2 - \ln(x^2 + 1)) - 2 \ln(x^2 + 1) + C$$

Multiplying out the brackets and gathering together terms in $\ln(x^2 + 1)$, this becomes:

$$\frac{1}{2}x^2 - \frac{1}{2}\ln(x^2 + 1) - 2\ln(x^2 + 1) + C = \frac{1}{2}x^2 - \frac{5}{2}\ln(x^2 + 1) + C$$

We can go one step further and factorise out the common factor of $1/2$ to arrive at this expression:

$$I_1 = \frac{1}{2}(x^2 - 5\ln(x^2 + 1)) + C$$

All that remains is to put this solution back into our earlier integration by parts formula to give the solution to our original integral as:

$$I = (x^3 - 4x)\tan^{-1}x - \frac{1}{2}(x^2 - 5\ln(x^2 + 1)) + C$$

Notes

We have examined these two examples in order to get a more detailed picture of the effect of our **LIATE** rule when we are faced with integrating a function of the form $P(x)f(x)$, where $P(x)$ is a polynomial in x and $f(x)$ is a (more or less complicated) logarithmic or inverse trigonometric function.

Our rule for choosing parts recommends that in these cases u should be chosen to be the logarithmic or inverse trigonometric function. Having made this choice, we find that the integral on the right-hand side of the integration by parts formula requires us to integrate a rational function in x . (This is because the logarithmic or inverse trigonometric function which we chose to be u differentiates to give a du/dx which is itself a rational function.) Multiplying this by the value we obtain for v by integrating dv/dx (which in these examples has been a straightforward, easily integrable, polynomial) provides an integrand with polynomials in the numerator and denominator. These are the integrals we have named I_1 in both the above examples.

These I_1 integrals can be split up into two parts, each of which has a single term in x^n , for some value of n , in the numerator. We have been able to integrate these separate parts fairly easily above, either by quoting results we obtained elsewhere in this book, or by a fairly simple application of one of our integration techniques.

The integration of the separate parts of I_1 , in both the above cases, required us to be able to integrate functions of the form:

$$\frac{x^n}{x^2 + 1}$$

where n has taken the values 1, 2 or 3. I have listed the integrals of the three rational functions which result in [Appendix 4 Section 2](#), along with some brief notes on how they are evaluated at various points in this book. If you find yourself regularly integrating functions of the type we have looked at here, then making use of these as “standard” integrals could save a lot of reworking.

I have also carried out a short investigation into the behaviour of these integrals for some values of n greater than 3 in [Appendix 5 Section 9](#).

8 More About Logarithmic and Inverse Trigonometric Functions

Before we summarise the conclusions we can draw from our work so far about the process of integration by parts, I will turn aside briefly to discuss a useful way of applying integration by parts to a group of functions which might not at first seem amenable to this technique. I will examine how we can use the technique to integrate two very simple functions which, in themselves, seem to defy the armoury of integration techniques we have so far accumulated. They can be made to respond by using what may seem at first to be a piece of mathematical conjuring. Looked at from another angle, however, this “trick” is simply a way of restating the problem in order to be able to apply integration by parts.

The two integrals I will look at here will also allow us to add to our list of standard integrals in [Appendix 4 Section 2](#), and to extend slightly the class of functions whose integration we can approach by parts.

Example 1

We will first address the following integral:

$$I = \int \ln x \, dx$$

Now, clearly, this integrand does not look like the product of two functions – and we have so far only used integration by parts on functions which are products – the division into parts itself requires us to allocate the two factors of the functional product to the two parts in the formula we have been using.

We can, however, express it (“trivially”, to use a favourite description of mathematicians for an action which appears rather simple and self-evident) as a product in the following way:

$$\ln x = 1 \times \ln x$$

Having done this, we can additionally point out that the function $f(x) = 1$ may be considered as an “algebraic” function for the purposes of our **LIATE** rule. It is in fact a “trivial” (that word again) polynomial, comprising a constant term only which has a coefficient of 1.

We can therefore restate our integral as:

$$I = \int 1 \cdot \ln x \, dx$$

The **LIATE** rule now dictates the following choice of parts:

$$u = \ln x \text{ and } \frac{dv}{dx} = 1$$

Integration by Parts

From this we can deduce:

$$\frac{du}{dx} = \frac{1}{x} \text{ and } v = x$$

and the integration by parts formula gives us the following:

$$\int 1 \cdot \ln x \, dx = x \ln x - \int \frac{1}{x} \cdot x \, dx$$

The integral on the right-hand side reduces (by cancellation) to:

$$\int 1 \, dx = x + C$$

so that our final solution can now be written out as:

$$I = x \ln x - x + C$$

Example 2

As our second example, we will evaluate the following integral, equally simple in appearance:

$$I = \int \tan^{-1} x \, dx$$

Using the same reasoning as we gave in [Example 1](#), we can restate this integral as:

$$I = \int 1 \cdot \tan^{-1} x \, dx$$

We now make the following choice of parts, as usual following the dictates of the **LIATE** rule:

$$u = \tan^{-1} x \text{ and } \frac{dv}{dx} = 1$$

These choices give us the following consequences:

$$\frac{du}{dx} = \frac{1}{x^2 + 1} \text{ and } v = x$$

The integration by parts formula then gives us this equation:

$$\int 1 \cdot \tan^{-1} x \, dx = x \tan^{-1} x - \int \frac{1}{x^2 + 1} \cdot x \, dx$$

The integral on the right-hand side, when written out in a more recognisable form, is one we have already integrated (by substitution), and it appears in [Appendix 4 Section 2](#) as a standard integral with the reference to the part of this book which explains the manner of its solution. It evaluates as follows:

$$\int \frac{x}{x^2 + 1} dx = \frac{1}{2} \ln(x^2 + 1) + C$$

We can therefore now write out our final solution as:

$$I = x \tan^{-1} x - \frac{1}{2} \ln(x^2 + 1) + C$$

Notes

We can now see that both of these integrals turned out to be fairly straightforward, once we rewrote them as the integral of a function multiplied by a factor of 1. Since we pointed out earlier in this section that $f(x) = 1$ should be regarded as an “algebraic” function, and the **LIATE** rule says that logarithmic or inverse trigonometric functions should be chosen as u in preference to algebraic functions when we are choosing parts, taking this approach means that we now only have to *differentiate* our logarithmic or inverse trigonometric function, rather than having to try to integrate it (ie it manages to transform a difficult integration operation into a simpler differentiation operation). These functions have readily obtainable derivatives, which are in fact rational functions; therefore, when multiplied by x , which is the value for v when $dv/dx = 1$, they give us a rational function to integrate on the right-hand side of the integration by parts formula.

9 An Algorithm for Integration by Parts

We ought now to be in a position to write down an algorithm for the process of integration by parts, based on the experience we have gained in the range of examples we have considered so far. I will first cover the “basic” technique, and then add a further algorithm for the tabular approach.

The “Basic” Technique

Step 1: Choose Parts u and dv/dx . We have applied integration by parts to integrands which are the product of two functions of x . The integration by parts formula which we use requires us to express our integrals in the form:

$$\int u \frac{dv}{dx} dx$$

One of the functions in our product will therefore have to be chosen as u , and the other will have to be chosen as dv/dx . We have described a simple rule for choosing parts when the two functions in our product are of different types: this rule is based on the following table of types of function:

Letter	Meaning	Examples
L	Logarithmic	$\ln x, \log_a x$

Integration by Parts

I	Inverse trigonometric	$\sin^{-1} x, \tan^{-1} x$
A	Algebraic	$x^2, 6x^4$
T	Trigonometric	$\cos x, \tan x$
E	Exponential	$e^x, 2^x$

We choose u to be the function in our product which comes in the higher position in this table; we choose dv/dx to be the other function. The rule for choosing parts can be abbreviated using the set of letters **LIATE**, as shown in the first column of the table.

The essential consideration when choosing parts is that the function chosen as u should be easy to differentiate, and that the function chosen as dv/dx should be easy to integrate. (The description “easy” is, of course, relative.) It is this consideration which forms the basis of the **LIATE** rule, and this consideration must be applied explicitly on a function-by-function basis in cases where both functions in the product are of the same type (such as [Section 4 Example 3](#)). In the particular cases where we have an integrand which is the product of an algebraic function and a logarithmic or inverse trigonometric function, we should choose u to be the logarithmic or inverse trigonometric function, since each of these will differentiate to give a rational function in x , and this will immediately reduce the difficulty of the task of integrating the whole integrand.

Step 2: Evaluate du/dx and v . The next step is to differentiate the function chosen as u in order to obtain du/dx . We must also integrate dv/dx in order to obtain a value for v . The requirement to obtain these two functions explains why the function chosen as u needs to be differentiable in a practical way, and that chosen as dv/dx must respond without too much difficulty to integration.

If one or other of these results cannot be easily obtained, then we may have made an incorrect choice of parts. In this situation, we should return to Step 1 above and make the alternative choice of parts. (Note that only two choices of parts are possible, since we only have two functions to choose from.)

If the alternative choice of parts does not make our task any easier, then we will be forced to the conclusion that our original integrand is not susceptible to integration by parts, and proceed to apply some other technique.

Step 3: Apply the Formula. Having obtained the two results called for in the previous step, we can now proceed to apply the integration by parts formula, which is as follows:

$$\int u \frac{dv}{dx} dx = uv - \int \frac{du}{dx} v dx$$

This formula allows us to express our original integral as the difference of the product of u and v and another integral, which is not identical to the one we started with. If we have chosen parts correctly and found that we can obtain all the values required by this formula, then we would hope to find that the integral on the right-hand side of the formula will be easier to evaluate than the one we started with. This should particularly be so if we have been able to differentiate u to produce a derivative which is *less complicated* than u itself, and if we have found that dv/dx can be integrated to give a function v which is at least *no more complicated* than dv/dx was.

We can see that these two conditions (which must *both* be fulfilled for the technique to work) together constitute a fairly strong limitation – first on the choice of parts which will be the more useful of the two possible ones, and second and more broadly on the type of integrand which will respond well to integration by parts at all.

Step 4: Evaluate the Resulting Right-Hand Side Integral. This step, which for simpler integrands is often the final step in the process, is to evaluate the integral which appears on the right-hand side of the integration by parts formula. This integral will be of the following form:

$$\int \frac{du}{dx} v dx$$

If we have made an appropriate choice of parts in Step 1, we should find that differentiating u and integrating dv/dx has led to a new product of functions which can be integrated more easily than the integrand we started with. We often aim to “differentiate away” the u component in some way – that is, we aim for a derivative of u which is *simpler* than u itself. In the cases where u is a polynomial in x - especially a simple one such as x^3 , or even just x itself, then differentiation will lead to a lower power of x than the one we began with. In the cases we looked at in [Section 7](#) above, where u is a logarithmic or inverse trigonometric function, then it will differentiate to give us a rational function of some sort – and this will often be easier to deal with than the function u itself.

In the case of the dv/dx part of our original integrand, this will have been integrated to produce v . If dv/dx has been chosen (using the **LIATE** rule) to be an exponential or trigonometric function, then integrating it will give rise to a function which is broadly of the same type. Even if it is a polynomial in x (as it was in the examples in [Section 7](#)), the worst that will happen is for us to be left with a polynomial in which each term is of a higher power than before – and this may not produce any additional complications in itself.

Whatever the integral on the right-hand side turns out to be, generally one of four distinct cases will arise. These, and the appropriate action in each case, are as follows:

Case 1: The integral can be integrated simply. If the integral we obtain can be evaluated using the basic rules of integration, then we simply evaluate it, and the right-hand side of the formula then gives us our final solution.

Case 2: The integral requires a specific technique. Sometimes (as we have seen in some of our recent examples) the integral we obtain requires either algebraic rearrangement or the use of another technique, such as substitution. Occasionally, both of these requirements are present. In this case, we simply proceed to apply whatever method we need in order to evaluate the integral, after which we will have a final solution.

Case 3: The integral must be integrated by parts. It is often the case that one application of the integration by parts technique simply delivers a right-hand side integral which must again be integrated using the same technique. We saw several examples of this kind in [Sections 5 and 6](#) above. In these cases, repeated integration by parts will often “differentiate away” one component of the original integrand to a constant factor, which can then be brought outside the integration sign to make the last integration step quite simple. In these cases, we simply

continue to apply integration by parts until we arrive at a right-hand side integral which can be integrated in one of the ways described under cases 1 and 2 above. Finally we put the results of each stage together to obtain our final solution.

Case 4: The original integral reappears. With some integrands, we will find that, after one or more applications of integration by parts, the original integrand itself appears again on the right-hand side of the formula. We have not encountered this behaviour so far, but we will examine some cases of this kind, and explain how to deal with them, in a later section below.

A Useful Quick Method

You may encounter the following “quick method” for creating the right-hand side of the integration by parts formula – it’s often quoted as a useful aid in carrying out the process entirely “in your head” – ie without explicit notes of the kind I have presented in the examples above. You can judge for yourself whether it helps or not.

The method runs as follows: first (using the **LIATE** rule or some other appropriate reasoning) divide the integrand into two factors (named u and dv/dx in the approach I have taken here) – these can be referred to as “the first part” (u) and “the second part” (dv/dx). We will create the right-hand side of the formula as two distinct terms.

To create the first term, we integrate the second part of our integrand (obtaining v from dv/dx), and multiply it by the first part to obtain the term uv .

We then add a minus sign and create the second term, which takes the form of an integral. The integrand is found by differentiating the first part (obtaining du/dx from u) and multiplying it by the integral of the second part (v), which we have already obtained in creating the first term.

We then deal with the effect of any minus sign arising from the second term, which may cause us to change the minus sign in the right-hand side expression to a plus sign, and the job is done.

It may help to reiterate this mental method without my explanatory notes linking it to the approach we have been using previously.

To obtain the first term, just integrate the second part, and multiply it by the first part. To obtain the integrand in the second term, differentiate the first part and multiply it by the integral of the second part.

The Tabular Method

Step 1: Choose Parts u and dv/dx . This step is identical to the first step in the previous subsection dealing with what I have called the “basic” method of integration by parts. Once again, we must express the integral we want to evaluate in the form:

$$\int u \frac{dv}{dx} dx$$

and choose the two parts u and dv/dx according to the **LIATE** rule.

Step 2: Create the Blank Table. We now create a table with three columns - headed u , v and $Sign$, as follows:

u	v	$Sign$

In the heading of the v column, we add a note of the part of our integrand we have chosen to be dv/dx , as in the following example:

u	$v (dv/dx = \sin x)$	$Sign$

Step 3: Complete the Table. We now proceed to complete as many rows of the table as we need to finish the job.

The first row. In the first column, we enter the value of u which results from our initial choice of parts. In the second column, we enter the first of our values for v - this first value is obtained by integrating the function we chose to be dv/dx when we chose parts. In the $Sign$ column, we place a minus sign.

Subsequent rows. For each subsequent row, we obtain the u entry by differentiating the u entry in the preceding row. We obtain each subsequent v entry by integrating the v entry in the preceding row. Each subsequent entry in the $Sign$ column is found by inverting the sign in the preceding row, so that if the preceding row contains a minus sign, we insert a plus sign; if the preceding row contains a plus sign, we insert a minus sign.

The last row. We know we have reached the last row in any particular table when the u entry, found by differentiating the u entry from the preceding row, is found to be zero. This will happen when the preceding u entry was a constant. Once we have encountered a u entry of zero, the table is finished, and we don't even need to fill in the other two columns of the final row, since these would contribute nothing to our final solution.

In all the integration steps we make while completing out table, we omit the constant of integration – all these constants will be amalgamated into one when we write out the final solution.

Step 4: Read off the Solution. We can now read off the final solution very simply from the rows of the completed table. For each row in turn, starting with the first, we multiply together the functions in the u and v columns and follow this product with the sign in the third column headed $Sign$. When we finally come to the last row, which should simply contain a u value of zero, we can of course drop the sign obtained from the third column of the preceding row, since there is nothing to follow it. All that remains is to add the constant of integration C .

You will find an abbreviated version of both these algorithms, without so much explanatory detail, in [Appendix 1 Section 2](#).

10 When Does Integration by Parts Work?

We can now put together some first thoughts on the question of when integration by parts can be expected to work well in evaluating particular integrals.

First of all, we can conclude from the various examples we looked at in [Section 4](#) above that functions of the form $xf(x)$, where $f(x)$ is an exponential, trigonometric, inverse trigonometric or logarithmic function, can be expected to respond well to integration by parts. We must, of course, remember to apply the **LIATE** rule in making our choice of parts, and this was demonstrated in the last two examples in [Section 4](#). (I will leave out of consideration here the situation we looked at in [Example 3 of Section 4](#), where both functions in the product were algebraic, since many of these integrands can be dealt with by substitution or with the help of some initial algebra.)

In [Sections 5 and 6](#), we found that functions of the form:

$$x^k f(x)$$

at least where $f(x)$ is trigonometric or exponential, could be integrated by applying our technique repeatedly, until the power of x had been “differentiated away” by repeated differentiation operations. The same approach could be made to integrands of the form:

$$P(x)f(x)$$

where $P(x)$ is a polynomial in x , and $f(x)$ once again trigonometric or exponential: either the polynomial would be “differentiated away”, producing an integrand with a polynomial of lower order at each stage, or the polynomial could be multiplied out, resulting in a series of integrands containing a power of x multiplied by $f(x)$, each of which could be integrated by parts separately.

In [Section 7](#), after a presentation of the **LIATE** rule for choosing parts, we saw two examples of functions of the form:

$$P(x)f(x)$$

where $f(x)$ was logarithmic or inverse trigonometric. These examples were dealt with (applying the **LIATE** rule) by choosing parts so that $f(x)$ was the part to be differentiated. Since the derivatives of logarithmic and inverse trigonometric functions are rational functions (of varying degrees of complexity), this led to a right-hand side integral which required the integration of a rational function. However many terms the polynomial $P(x)$ may contain, we would expect this latter task to respond eventually to some algebraic rearrangement, rendering the integration practical.

Finally, in [Section 8](#), we used an interesting device to make logarithmic and inverse trigonometric functions themselves respond to integration by parts. We did this by rewriting the integral in each case as follows:

$$\int f(x)dx = \int 1 \cdot f(x)dx$$

By choosing parts (again applying the **LIATE** rule) so that:

$$u = f(x) \text{ and } \frac{dv}{dx} = 1 \text{ (so that } v = x)$$

the integration by parts formula gave us a solution that looked like this:

$$xf(x) - \int xf'(x)dx$$

Because of the nature of the derivatives of logarithmic and inverse trigonometric functions which I mentioned a few lines ago, this gives us a relatively straightforward rational function to integrate on the right-hand side.

We now have some idea of two classes of integrand for which integration by parts will work.

11 Definite Integrals Using Integration by Parts

An interesting and important question arises from the integration by parts formula when we ask ourselves how we evaluate definite integrals when using this technique. The question arises because the right-hand side of the integration by parts formula contains a first term which is not itself an integral, while at the same time not being the complete solution to any indefinite integral. Therefore it is not immediately clear how to treat the right-hand side of the formula when the left-hand side is a definite integral.

We can, however, reason out the answer to this question without serious difficulty. In order to do this, I will introduce some simple notation to make things easier.

Let us consider a general integral which is to be evaluated by parts in the way which we have done throughout this book. We will call its solution $F(x)$. We therefore have:

$$F(x) = \int u \frac{dv}{dx} dx$$

The integration by parts formula gives us the following identity:

$$\int u \frac{dv}{dx} dx = uv - \int \frac{du}{dx} v dx$$

We will call the solution to the indefinite integral on the right-hand side of this formula $G(x)$. Thus we have:

$$G(x) = \int \frac{du}{dx} v dx$$

Now, when looking at definite integrals using integration by parts, we will be dealing with integrals of the form:

Integration by Parts

$$\int_a^b u \frac{dv}{dx} dx$$

Using the notation we have decided upon, we can see that this is equal to:

$$F(b) - F(a)$$

Additionally, if we recall that the items named u and v in the integration by parts formula in fact represent functions $u(x)$ and $v(x)$, we can rewrite that formula with our chosen notation as follows:

$$F(x) = u(x)v(x) - G(x)$$

These last two statements combine to allow us to represent our general definite integral as follows:

$$\int_a^b u \frac{dv}{dx} dx = F(b) - F(a) = (u(b)v(b) - G(b)) - (u(a)v(a) - G(a))$$

The final expression above can be rearranged like this:

$$(u(b)v(b) - u(a)v(a)) - (G(b) - G(a))$$

and this can be expressed using the following notation:

$$\left[u(x)v(x) \right]_a^b - \int_a^b \frac{du}{dx} v dx$$

The square-bracket notation applied to $u(x)v(x)$ is to be interpreted in the same way as that used when evaluating definite integrals, for example as follows:

$$\int_a^b f(x) dx = \left[F(x) \right]_a^b = F(b) - F(a)$$

Returning to our main line of reasoning, then, we are now able to conclude that, when evaluating a definite integral by parts:

$$\int_a^b u \frac{dv}{dx} dx = \left[uv \right]_a^b - \int_a^b \frac{du}{dx} v dx$$

I have removed the bracketed x notation from the uv term on the right-hand side in order to reflect more closely the most common version of the integration by parts formula.

12 Integration by Solving an Equation (When the Integrand Just Won't Go Away)

In [Section 9](#), where we set out an algorithm for the “basic” method for doing integration by parts, the final step of our algorithm foreshadowed a situation which we have not so far encountered – that is when the integral on the right-hand side of the integration by parts formula is found to contain the same integrand we originally started with. It is now time to examine some integration problems where this behaviour occurs, and to illustrate the fairly simple way in which we can deal with such problems. As usual, we will do this by examining a series of examples.

Example 1

We will look in this example at the integration of a product of two functions unlike those we have seen so far. Our integral will be the following:

$$I = \int e^x \sin x \, dx$$

This is, of course, the product of an exponential function and a trigonometric one. Let's try to approach its evaluation using integration by parts, starting with the following choice of parts based on the **LIATE** rule:

$$u = \sin x \text{ and } \frac{dv}{dx} = e^x$$

These choices give us the following consequences:

$$\frac{du}{dx} = \cos x \text{ and } v = e^x$$

and from this, the integration by parts formula lets us conclude that:

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$$

We can already see that this application of our technique has given us a right-hand side integral which is not so different from the original integral I . Let us not despair, however, and investigate a little further by setting:

$$I_1 = \int e^x \cos x \, dx$$

We will now see what happens when we try integrating this right-hand side integral by another application of integration by parts.

Sticking with the **LIATE** rule, we make the following choice of parts:

$$u = \cos x \text{ and } \frac{dv}{dx} = e^x$$

These choices lead to these consequences:

$$\frac{du}{dx} = -\sin x \text{ and } v = e^x$$

The integration by parts formula thus gives us the following result for the value of I_1 :

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx$$

(Note that the right-hand side expression contains a plus sign: this is because we have taken the minus sign in front of the value of du/dx outside the integral sign, and combined it with the usual integration by parts minus sign to give a resulting plus.)

We immediately notice here that we seem to have gone round in a circle – the integral on the right-hand side of the expression for I_1 is precisely the integral I , which we wanted to evaluate in the first place.

So have we actually achieved anything at all? In fact, all is by no means lost, as we will see when we start putting all these results together.

First, we recall that:

$$I = \int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx = e^x \sin x - I_1$$

from our first application of integration by parts. Substituting into this equation the expression we obtained for I_1 from our second integration by parts, we obtain the following expression for I :

$$I = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

Recalling that the integral which we are left with on the right-hand side is precisely our original integral I , this gives us the following equation :

$$I = e^x \sin x - e^x \cos x - I$$

Now, this is actually a simple equation in I which we can solve using elementary algebra, just as if it were the equation:

$$x = a - b - x$$

This equation in x could be simply rearranged to give us the following version:

$$2x = a - b$$

from which we would be able to conclude that:

Integration by Parts

$$x = \frac{1}{2}(a - b)$$

Let's apply exactly the same approach to the equation in I which we derived after two applications of integration by parts. A first rearrangement gives:

$$2I = e^x \sin x - e^x \cos x$$

and we can solve this very simply to give us a final solution of:

$$I = \frac{1}{2}e^x(\sin x - \cos x) + C$$

(We have done two additional things in moving from the expression for $2I$ to this final expression for I : we have factorised out the common factor of e^x from the two terms in the $2I$ equation, and we have finally taken care to add in the constant of integration C , bearing in mind that this might in practice be the combination of constants of integration from the two integration operations which led to our final equation in I .)

Note

It's worth pointing out clearly that what might at first sight have seemed a failure of the integration by parts approach in this case proved in the end to be no such thing. We did find, after two applications of the technique, that the original integral I made its reappearance on the right-hand side of the expression we built up. However, we treated this as an equation with instances of I on both sides, which we were then able to rearrange to obtain a final expression for I .

As it turned out, two applications of integration by parts had given us enough material from the uv terms alone of the two instances of the formula to provide us with a solution to our original integration problem.

Example 2

In order to get a better sense of how this process works, let's look at a similar but slightly more complicated integral.

$$I = \int e^{2x} \cos 3x \, dx$$

Our approach will follow very similar lines to that of [Example 1](#). We will start by choosing parts according to the **LIATE** rule, as follows:

$$u = \cos 3x \quad \text{and} \quad \frac{dv}{dx} = e^{2x}$$

These choices will lead to the following consequences:

$$\frac{du}{dx} = -3 \sin 3x \quad \text{and} \quad v = \frac{1}{2}e^{2x}$$

Integration by Parts

The integration by parts formula then gives:

$$\int e^{2x} \cos 3x \, dx = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{2} \int e^{2x} \sin 3x \, dx$$

Once more, we have an integral on the right-hand side which bears some similarity to I . We can proceed to solve this, naming it:

$$I_1 = \int e^{2x} \sin 3x \, dx$$

We will choose parts in a similar way to the approach we took when choosing parts for the original integral I :

$$u = \sin 3x \text{ and } \frac{dv}{dx} = e^{2x}$$

These choices give us the following:

$$\frac{du}{dx} = 3 \cos 3x \text{ and } v = \frac{1}{2} e^{2x}$$

From the integration by parts formula:

$$I_1 = \frac{1}{2} e^{2x} \sin 3x - \frac{3}{2} \int e^{2x} \cos 3x \, dx$$

And so we see once again that the original integral I has made its reappearance on the right-hand side of the expression for I_1 .

Substituting this expression into the result of the integration by parts formula as applied to the original integral I will give us the following:

$$I = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{2} \left(\frac{1}{2} e^{2x} \sin 3x - \frac{3}{2} I \right)$$

Multiplying out the material in brackets, we can turn this into:

$$I = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x - \frac{9}{4} I$$

Rearranging in order to make I the subject of this equation give us as a first step:

$$\frac{13}{4} I = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x$$

from which we can obtain:

$$I = \frac{2}{13}e^{2x} \cos 3x + \frac{3}{13}e^{2x} \sin 3x$$

One last step of factorisation allows us to reach the following solution to our integration problem:

$$I = \frac{1}{13}e^{2x}(2 \cos 3x + 3 \sin 3x) + C$$

As we might expect, this solution is not dissimilar in shape to the one we obtained in [Example 1](#). The additional arithmetic we had to go through arises from the factors of x which are present in the original integrand. However, the appearance of the unusual fractional coefficient of $1/13$ is an interesting phenomenon which we note in passing.

(Having noted this phenomenon, readers interested in finding out more about the characteristics of integrals whose integrands are products of exponential and trigonometric functions may like to examine the evaluation of the integral of the general function $e^{ax} \cos bx$ which I give in [Appendix 5 Section 10](#).)

Example 3: “Aargh! Sec Cubed!”

In this example, we will look at what is sometimes regarded as a “notable” integral, whose solution by parts offers an interesting example of the behaviour we are looking at in this section. The integral is:

$$I = \int \sec^3 x dx$$

(The exclamation which I have quoted in the title of this example was uttered by one of my old university tutors when he was confronted with this integral in the course of solving a problem in mechanics. It therefore gives me great pleasure to be able to present a simple and satisfying solution – which, sadly, eluded me and all my contemporaries at the time - in these pages!)

In order to integrate by parts, we need our integrand to be expressed as the product of two functions. This can be achieved here by rewriting our integral as follows:

$$I = \int \sec x \cdot \sec^2 x dx$$

We are now faced with the question of how to choose parts. The **LIATE** rule will be of no assistance here, since both parts of the integrand are of the same type as far as this rule is concerned – both are trigonometric functions. However, if we are able to recognise $\sec^2 x$ as the derivative of $\tan x$, then we can start by making the following choices:

$$u = \sec x \text{ and } \frac{dv}{dx} = \sec^2 x$$

This leads to the following conclusions:

Integration by Parts

$$\frac{du}{dx} = \sec x \tan x \text{ and } v = \tan x$$

The integration by parts formula then produces the following result:

$$\int \sec^3 x dx = \sec x \tan x - \int \sec x \tan^2 x dx$$

We can probably be forgiven for thinking that the integral on the right-hand side is no easier to evaluate than the one we started with; however, when confronted with a complicated arrangement of trigonometric functions, we should always be prepared to see if one of the rich set of trigonometric identities might come to our aid. In this case, we might recall the following identity (which is among those listed in [Appendix 4 Section 3](#)):

$$\tan^2 x = \sec^2 x - 1$$

This allows us to represent the squared tangent in our right-hand side integral in terms of the secant function. Does it get us anywhere? Well, we can now restate the right-hand side integrand as follows:

$$\sec x \tan^2 x = \sec x (\sec^2 x - 1) = \sec^3 x - \sec x$$

Paying close attention to the astonishing piece of manipulation which now follows (there is nothing up my sleeve), we see that the integration by parts result for our integrand I can now be rewritten as:

$$\int \sec^3 x dx = \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$$

I has now reappeared on the right-hand side! Indeed, we can now form the equation in terms of I which we have been producing in the previous examples in this section. This will be as follows:

$$I = \sec x \tan x - I + \int \sec x dx$$

Rearranging to take all the instances of I to the left-hand side and evaluating the integral of $\sec x$ (which is achieved by substitution in [Appendix 5 Section 6](#)), we will have the following:

$$2I = \sec x \tan x + \ln|\sec x + \tan x|$$

A final step to make I the subject will give the following solution to our integral:

$$I = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C$$

Notes

What special points of technique did this integration problem call for? Apart from the use of integration by parts to produce an equation in I which we then proceeded to solve by algebraic rearrangement, there were only two: first, splitting up the integrand into a product of two functions which allowed us to observe that $\sec^2 x$ is the derivative of $\tan x$ and choose parts accordingly; and second, using the trigonometric identity for $\tan^2 x$ to express the right-hand side integrand in terms of secant functions alone.

It's also worth noting that, unlike the situation in [Examples 1 and 2](#) of this section, we only had to apply integration by parts once in this case to reveal the presence of an instance of I on the right-hand side of the formula.

The approach we took here turns out to be quite useful in integrating higher powers of $\sec x$ by reducing the problem to the integration of a lower power. An investigation of this approach is made in [Appendix 5 Section 11](#).

Finally, this example might give us the idea that integration by parts could be usefully applied to integrands which can be rewritten as the product of two functions, where one is recognisable as the derivative of another function. This becomes especially useful in situations where the derivative concerned is more complicated than simply an instance of an elementary function. In the case of the current example, the derivative we were able to recognise was $\sec^2 x$; we might expect to be able to apply the same technique where the integrand is the product of some function and $\sec x \tan x$, which is the derivative of $\sec x$. Once again, [Appendix 5 Section 12](#) contains one example of the evaluation of such an integral.

Example 4: It Doesn't always Work

It must be pointed out, before we become wild with enthusiasm about the possibilities of this slightly more sophisticated application of integration by parts, that *it doesn't always work!* Sometimes, we obtain an equation in I which either cannot be solved, or that when solved produces the useless result that $I = 0$. We can illustrate this kind of behaviour by approaching the following integral, and watching carefully:

$$I = \int \sin 3x \cos 5x \, dx$$

(We might pause a moment to note that this integrand bears some similarities to the one which was the basis of the opening example in [Part 1](#) of this book, where we evaluated it using substitution. In that example, however, we were able to spot a function-derivative pair, the presence of which identifies an integral as ideal for the method of substitution. No such pair exists here, however, due to the different coefficients of x in the two trigonometric functions. Additionally, we have not looked in this part of the book at very many examples of products of two functions of the same type – apart, of course, from [Example 3](#) above. However, trying to integrate this example by parts will produce a very interesting result, which will be reason enough to continue.)

The first thing we will notice is that the **LIATE** rule doesn't help us choose parts here. However, if we take the view that things are often easier if we avoid introducing minus signs,

Integration by Parts

we will note that $\sin 3x$ differentiates without introducing a minus sign, and that $\cos 5x$ integrates without bringing in a minus sign. Our choice, informed by these observations, will therefore be:

$$u = \sin 3x \text{ and } \frac{dv}{dx} = \cos 5x$$

The consequences of these choices are:

$$\frac{du}{dx} = 3 \cos 3x \text{ and } v = \frac{1}{5} \sin 5x$$

The integration by parts formula then gives us:

$$\int \sin 3x \cos 5x \, dx = \frac{1}{5} \sin 3x \sin 5x - \frac{3}{5} \int \sin 5x \cos 3x \, dx$$

Now, in these circumstances – when we have another integral which cannot be integrated using our elementary rules – we have by now become used to proceeding with a further application of integration by parts. Accordingly, we will set:

$$I_1 = \int \sin 5x \cos 3x \, dx$$

and, by analogy with our first step, we will make the following choice of parts:

$$u = \sin 5x \text{ and } \frac{dv}{dx} = \cos 3x$$

The consequences of these choices are:

$$\frac{du}{dx} = 5 \cos 5x \text{ and } v = \frac{1}{3} \sin 3x$$

The integration by parts formula then gives us:

$$\int \sin 5x \cos 3x \, dx = \frac{1}{3} \sin 3x \sin 5x - \frac{5}{3} \int \sin 3x \cos 5x \, dx$$

As we have come to expect in this section, the original integral I has now reappeared on the right-hand side. Rewriting this equation in terms of our I -notation, we will obtain:

$$I_1 = \frac{1}{3} \sin 3x \sin 5x - \frac{5}{3} I$$

If we now substitute this expression for I_1 into the result of our first application of integration by parts, we obtain the following equation in I :

$$I = \frac{1}{5} \sin 3x \sin 5x - \frac{3}{5} \left(\frac{1}{3} \sin 3x \sin 5x - \frac{5}{3} I \right)$$

Multiplying out the brackets, we arrive at the following:

$$I = \frac{1}{5} \sin 3x \sin 5x - \frac{1}{5} \sin 3x \sin 5x + I$$

The first two terms cancel each other out, and we are left with the supremely uninformative equation:

$$I = I$$

which we can quite reasonably deem both tautologous and trivial.

When our chosen technique either fails completely or leads us to a useless result of this kind, we must conclude that we have chosen the wrong technique – and indeed this is the appropriate conclusion if integration by parts produces an equation in I which either cannot be solved or contains no useful information (as in the current example).

One observation which arises from what we have seen here is this: integration by parts has worked well for integrands which were the products of two *different* types of function. This is because the different behaviour of the two factor functions under differentiation and integration is what allows us to arrive at a *simpler* integral on the right-hand side of each application of the integration by parts formula. The examples in this section have thrown up the original integrand I once more on the right-hand side. (This is sometimes described as the original function “*remaining in-house*”.) Sometimes, we were able to solve the resulting equation in order to arrive at a value for I . In the current example, where the functions in the product were of the same type, their behaviour under differentiation and integration led us to a useless and uninformative equation.

Integration by parts is therefore not appropriate for this particular integral. Those who might like to know my recommended approach to this example will find a fairly stress-free solution using trigonometric formulae in the first example in [Part 3 Section 9](#).

Notes

Can we say anything about the circumstances in which this method of integration might or might not be expected to work? I offer the following few observations.

In [Examples 1 and 3](#) above, we used integration by parts to obtain the following equation for our integral I :

$$I = f(x) - I \text{ so that } 2I = f(x) \text{ and } I = \frac{1}{2}f(x)$$

As we can see, this type of equation can be easily solved, and we can obtain from it a suitable expression for the solution to our integral I .

[Example 2](#) gave us a slightly different kind of equation – it had this form:

$$I = f(x) - \frac{p}{q}I \text{ so that } \left(1 + \frac{p}{q}\right)I = f(x) \text{ or } \frac{p+q}{q}I = f(x) \text{ and so } I = \frac{q}{p+q}f(x)$$

Once again, we can solve this kind of equation and obtain a solution from it.

We can, of course, go a little further, and conclude by the same kind of reasoning that a similar equation containing a plus sign instead of the above minus sign can also be solved. We can see this as follows:

$$I = f(x) + \frac{p}{q}I \text{ gives us } \left(1 - \frac{p}{q}\right)I = f(x) \text{ or } \frac{q-p}{q}I = f(x) \text{ and so } I = \frac{q}{p-q}f(x)$$

(This conclusion could, of course, be derived by simply considering what would happen if either the numerator or denominator of the fractional coefficient p/q were to be negative.)

However, the following situation, which arises when we replace the minus sign in the first type of equation I described above with a plus sign, gives us one situation where our equation has no useful solution:

$$I = f(x) + I \text{ in general only has the solution } I = 0$$

and so an if an equation of this kind arises from the application of integration by parts, we will not get anything helpful from it, and will need to consider a different technique of integration.

Lastly, we should note the equation which arose from [Example 4](#): here we had an equation of the form:

$$I = f(x) + I \text{ where } f(x) = 0$$

We have already seen that such an equation, where the instance of I on the right-hand side is preceded by a plus sign, does not help us to solve the integral – in the case of [Example 4](#), the equation is rendered even more useless by the shape finally taken by the expression $f(x)$.

13 Reduction Formulae

We are now in a position to examine one of the most interesting consequences arising from use of the integration by parts technique. In order to approach it, we will start by looking at the integration of a higher power of a trigonometric function. We will start by considering the following integral:

$$I = \int \cos^7 x dx$$

We might start approaching this integral using the integration by parts technique, in a similar way to the approach we took to the integration of $\sec^3 x$ in [Section 12 Example 3](#) above – this would involve expressing the integrand as follows:

$$\cos^7 x = \cos x \cdot \cos^6 x$$

Integration by Parts

Doing this enables us to make a useful choice of parts: $\cos x$ is easy to integrate, so it can be chosen as dv/dx ; $\cos^6 x$, on the other hand, is no easier to integrate than the original integrand $\cos^7 x$ - but it will respond very submissively to the Chain Rule for differentiation. We therefore choose it as u . To summarise, our choice of parts will therefore be:

$$u = \cos^6 x \text{ and } \frac{dv}{dx} = \cos x$$

The consequences of these choices are as follows:

$$\frac{du}{dx} = 6\cos^5 x (-\sin x) = -6 \sin x \cos^5 x \text{ and } v = \sin x$$

The integration by parts formula now gives us:

$$\int \cos^7 x dx = \sin x \cos^6 x + 6 \int \sin^2 x \cos^5 x dx$$

(Note that the usual minus sign produced by the integration by parts formula has been changed to a plus sign since dv/dx has a minus sign of its own - additionally, I have taken the liberty of moving the signed coefficient of -6 outside the integral sign.)

The integral on the right now begins to look a little disturbing. However, we can use a rearrangement of the Pythagorean identity which connects the squares of the sine and cosine functions to rewrite the $\sin^2 x$ factor as follows:

$$\sin^2 x = 1 - \cos^2 x$$

This lets us re-express the integral on the right-hand side as follows:

$$\int \sin^2 x \cos^5 x dx = \int \cos^5 x dx - \int \cos^7 x dx$$

We can see that, as happened several times in the preceding section, the original integral has now reappeared on the right-hand side. Substituting our results into the equation we obtained from the integration by parts formula above, we now have the following equation in I :

$$I = \sin x \cos^6 x + 6 \int \cos^5 x dx - 6I$$

We could now, of course, rearrange this equation to make I the subject. This would still leave us with the task of integrating $\cos^5 x$ in order to get a useful solution. By analogy with what we have just seen while integrating $\cos^7 x$, we could expect this to leave us with the additional necessity of integrating $\cos^3 x$ - which in turn would require similar working.

In order to shortcut this repetitive process (note that tabular integration by parts, at least in the form we have used it in earlier sections, might not be very useful here, since we expect to obtain an equation in I at each stage of the process rather than a straightforward solution involving nothing more than another integral to evaluate) we will try to look at the more general case, and consider the following integral:

Integration by Parts

$$I_n = \int \cos^n x dx$$

in order to see if we can derive some kind of general solution which might save some of the labour of repeated iterations. (We have given the letter representing our integral a subscript of n to indicate that it is the integral of the n -th power of $\cos x$; if we encounter integrals of lower powers – as we might expect to do, based on our experience so far – we will be able to name them simply and suggestively in this way.)

We will go about obtaining our general result using the same approach we took in our brief glance at $\cos^7 x$ a moment ago. We will start by splitting the integrand into two factors as follows:

$$\cos^n x = \cos x \cdot \cos^{n-1} x$$

Just as before, this will let us make the following choice of parts:

$$u = \cos^{n-1} x \text{ and } \frac{dv}{dx} = \cos x$$

This will lead to the following consequences:

$$\frac{du}{dx} = (n-1)\cos^{n-2} x (-\sin x) = -(n-1)\sin x \cos^{n-2} x \text{ and } v = \sin x$$

Accordingly, the integration by parts formula will give us:

$$\int \cos^n x dx = \sin x \cos^{n-1} x + (n-1) \int \sin^2 x \cos^{n-2} x dx$$

Using the same trigonometric identity as we used earlier, we can restate the integral on the right-hand side as follows:

$$\int \sin^2 x \cos^{n-2} x dx = \int (1 - \cos^2 x) \cos^{n-2} x dx = \int \cos^{n-2} x dx - \int \cos^n x dx$$

This can be substituted back into the result we obtained from the integration by parts formula; we can also name the various integrals we encounter using the subscript convention we have adopted here. This gives us the following equation:

$$I_n = \sin x \cos^{n-1} x + (n-1)(I_{n-2} - I_n)$$

Expanding out the brackets on the right-hand side, this becomes:

$$I_n = \sin x \cos^{n-1} x + (n-1)I_{n-2} - (n-1)I_n$$

Adding $(n-1)I_n$ to both sides, in order to move all the I_n terms to the left-hand side, we obtain:

Integration by Parts

$$nI_n = \sin x \cos^{n-1}x + (n-1)I_{n-2}$$

Finally, dividing by n , we arrive at the following:

$$I_n = \frac{1}{n} \sin x \cos^{n-1}x + \frac{n-1}{n} I_{n-2}$$

This equation, which expresses the integral of $\cos^n x$ in terms of the integral of a *lower* power of $\cos x$, is the *reduction formula* referred to in the title of this section. It reduces the power of $\cos x$ which we are called upon to integrate – if used repeatedly, it will eventually leave us with only the need to integrate $\cos x$ itself if n is odd, or just 1 (ie $\cos^0 x$) if n is even.

Let's note that we could just as easily write this reduction formula with the two terms in reverse order, as follows:

$$I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n} \sin x \cos^{n-1}x$$

This remark may seem to be both obvious and uninteresting; it will, however, become a little more relevant when come to the end of [Example 1](#) below.

Let's see this in action, in the hope that it might make it easier for us to integrate $\cos^7 x$, the problem with which this discussion began. We will apply the reduction formula we have derived to the integral:

$$I_7 = \int \cos^7 x dx$$

renaming our integral using the subscript notation we adopted when obtaining our reduction formula.

The reduction formula gives us the following equation:

$$I_7 = \frac{1}{7} \sin x \cos^6 x + \frac{6}{7} I_5$$

Substituting into this equation the result of applying the reduction formula once more, this time to I_5 , we get:

$$I_7 = \frac{1}{7} \sin x \cos^6 x + \frac{6}{7} \left(\frac{1}{5} \sin x \cos^4 x + \frac{4}{5} I_3 \right) = \frac{1}{7} \sin x \cos^6 x + \frac{6}{35} \sin x \cos^4 x + \frac{24}{35} I_3$$

Let's repeat this operation once more, this time substituting into our equation the result of applying the reduction formula to I_3 :

$$I_7 = \frac{1}{7} \sin x \cos^6 x + \frac{6}{35} \sin x \cos^4 x + \frac{24}{35} \left(\frac{1}{3} \sin x \cos^2 x + \frac{2}{3} I_1 \right)$$

Now, I_1 is no more than the integral of $\cos x$; in symbols:

$$I_1 = \int \cos x \, dx = \sin x + C$$

so we can put this back into the latest version of our expression for I_7 and take the final step of multiplying out the bracketed material to obtain the following final solution:

$$\int \cos^7 x \, dx = \frac{1}{7} \sin x \cos^6 x + \frac{6}{35} \sin x \cos^4 x + \frac{24}{105} \sin x \cos^2 x + \frac{48}{105} \sin x + C$$

Notes

This solution has the characteristic shape of expressions resulting from repeated use of integration by parts – a series of terms of varying complexity. The ever-more outrageous fractional coefficients are interesting: we can form some notion of how they arise, however, when we recall the coefficients $1/n$ and $(n-1)/n$ which appear in the reduction formula we obtained connecting I_n with I_{n-2} . These will be compounded repeatedly through the repeated uses we made of the formula in evaluating our integral.

The work involved only three applications of the reduction formula: to I_7 , I_5 and I_3 . Each of these applications was nothing more than the substitution into the formula of the appropriate value of n , and then the multiplying out of the bracketed material in each case.

There's little doubt that this is much less laborious than repeated integration by parts. We can therefore add another weapon to our arsenal of integration techniques: when dealing with the integration of a high integral power of some function which we can integrate easily when it is not raised to a power, it will save some labour to derive a reduction formula for the integral of the n -th power of our function, and then apply this formula to *reduce* successively the power of the function which we need to integrate. We eventually arrive at an easily integrated function, and all that is necessary is to add that to the series of terms we have built up by successive use of the reduction formula.

In the case we have examined above, we found that our reduction formula linked I_n to I_{n-2} , so that we were able to step down the powers two at a time. In our example, only three uses of the reduction formula were needed until we reached the integral of $\cos x$ alone.

While I have not attempted to set out a tabular arrangement for the solution given, I would point out that this repeated use of a reduction formula is in a sense not completely dissimilar to the tabular method of repeated integration by parts explained in [Section 6](#) above, although any table we constructed might look a little different from the tables of [Section 6](#).

Let's now explore this interesting technique further by means of a series of examples.

Example 1

Having looked at the integration of higher powers of the cosine function, it seems natural to be curious about higher powers of the sine. In order to assuage this curiosity, we will now examine the following general integral:

$$I_n = \int \sin^n x \, dx$$

Integration by Parts

After the detailed work we did on the opening example of this section, we should now feel confident enough to jump straight in and evaluate the general integral, rather than looking at one particular value of n .

Since we are dealing with an integrand very similar to the one we have just dealt with, we should expect to be able to proceed in exactly the same way. So we will start by rewriting our integrand as follows:

$$\sin^n x = \sin x \cdot \sin^{n-1} x$$

That said, we can now choose parts for our integration as follows:

$$u = \sin^{n-1} x \text{ and } \frac{dv}{dx} = \sin x$$

From these choices we can conclude that:

$$\frac{du}{dx} = (n-1)\sin^{n-2} x \cos x \text{ and } v = -\cos x$$

The integration by parts formula now gives us:

$$\int \sin^n x dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x dx$$

(Note that the minus sign in front of the value for v , when taken outside the integral sign, has transformed the usual minus sign we get from the integration by parts formula into a plus sign.)

The integral on the right-hand side contains a $\cos^2 x$ factor; just as we did with the earlier cosine integral, we can use the Pythagorean identity which links the sine and cosine to rewrite this integral as follows:

$$\int \sin^{n-2} x \cos^2 x dx = \int \sin^{n-2} x (1 - \sin^2 x) dx = \int \sin^{n-2} x dx - \int \sin^n x dx$$

This now lets us write out the result of our integration by parts using the subscripted- I notation we have adopted in the following way:

$$I_n = (n-1)(I_{n-2} - I_n) - \sin^{n-1} x \cos x$$

(Note that I have changed the order of terms in this equation so as to avoid the disagreeable presence of an opening minus sign.)

A couple of additional operations are called for in order to arrive at our final reduction formula. First, we multiply out our bracketed material to obtain:

$$I_n = (n-1)I_{n-2} - (n-1)I_n - \sin^{n-1} x \cos x$$

Adding $(n - 1)I_n$ to both sides will give us this:

$$nI_n = (n - 1)I_{n-2} - \sin^{n-1}x \cos x$$

A final division by n gives us the result we want:

$$I_n = \frac{n - 1}{n}I_{n-2} - \frac{1}{n}\sin^{n-1}x \cos x$$

As we might expect, apart from some differences of sign and a little swapping around between sine and cosine, this is quite similar to the reduction formula we obtained for the integral of $\cos^n x$ at the beginning of this section.

Reduction Formula for a Definite Integral

So far, we have derived reduction formulae for two general integrals; these have been indefinite integrals, and we have seen how to use the reduction formula we obtain to evaluate indefinite integrals for specific values of the variable n which appeared in the general integrand. It is also possible to derive reduction formulae for *definite* integrals, and these can be quite useful in practical applications. We will now look at an example of this process, with specific reference to the general integral of $\sin^n x$ which we have just examined.

We recall, first of all, that the result of applying integration by parts to the integral in [Example 1](#) above gave us the following equation:

$$\int \sin^n x dx = -\sin^{n-1}x \cos x + (n - 1) \left(\int \sin^{n-2}x dx - \int \sin^n x dx \right)$$

splitting up the right-hand side integral into the two components we obtained by using a trigonometric identity. [Section 11](#) above, where we looked at the how to deal with definite integrals when integrating by parts, tells us that in the case of a definite integral of $\sin^n x$, the above equation would look as follows:

$$\int_a^b \sin^n x dx = \left[-\sin^{n-1}x \cos x \right]_a^b + (n - 1) \left(\int_a^b \sin^{n-2}x dx - \int_a^b \sin^n x dx \right)$$

Let's now consider one particular definite integral and adopt a suitable naming convention. We will write:

$$J_n = \int_0^{2\pi} \sin^n x dx$$

using the letter J with a subscript in order to avoid confusion with the names we used for our indefinite integrals. Using this style of abbreviation, we can write down the following equation for our chosen definite integral:

Integration by Parts

$$J_n = \left[-\sin^{n-1}x \cos x \right]_0^{2\pi} + (n-1)(J_{n-2} - J_n)$$

Now, since $\sin 0$ and $\sin 2\pi$ are both equal to zero, the first term on the right-hand side is itself zero, and can be removed completely. We therefore have the following much simpler equation for our definite integrals:

$$J_n = (n-1)(J_{n-2} - J_n)$$

Just as we did with our reduction formula for I_n in [Example 1](#), we can multiply out the brackets to give:

$$J_n = (n-1)J_{n-2} - (n-1)J_n$$

subsequently bring all the terms in J_n to the left-hand side to obtain:

$$nJ_n = (n-1)J_{n-2}$$

and finally divide by n to arrive at the following simple and attractive reduction formula:

$$J_n = \frac{n-1}{n}J_{n-2}$$

This exhibits a simple and elegant relationship between J_n and J_{n-2} . In fact, we can go further and derive a direct formula for the definite integral J_n which does not depend on any other related integral. I have relegated discussion of the investigation and proof of such a formula to [Appendix 5 Section 13](#) so as to avoid interrupting our consideration of reduction formulae here.

Example 2

Let's now look at an example which does not contain any trigonometric functions. We will consider the following general indefinite integral:

$$I_n = \int x^n e^{2x} dx$$

As before, we will try to obtain a reduction formula for the indefinite integral, and then proceed to a reduction formula for a particular definite integral. We will finally apply this in order to evaluate a definite integral for a particular value of n .

The first remark we can make is that, unlike the powers of trigonometric functions we have been looking at so far in this section, the integrand in this example allows us to use the **LIATE** rule to choose parts for integration. Accordingly, we choose as follows:

$$u = x^n \text{ and } \frac{dv}{dx} = e^{2x}$$

The consequences are:

Integration by Parts

$$\frac{du}{dx} = nx^{n-1} \text{ and } v = \frac{1}{2}e^{2x}$$

and so the integration by parts formula gives us the following:

$$\int x^n e^{2x} dx = \frac{1}{2} x^n e^{2x} - \frac{n}{2} \int x^{n-1} e^{2x} dx$$

We can now rewrite this very simply using our subscripted- I notation as follows:

$$I_n = \frac{1}{2} x^n e^{2x} - \frac{n}{2} I_{n-1}$$

thus arriving at our reduction formula in one simple stage, and without any additional algebra.

Let's now have a look at one particular definite integral. We will choose:

$$J_n = \int_0^1 x^n e^{2x} dx$$

Looking back at the equation we obtained above from applying integration by parts to the indefinite integral I_n , we can easily write down the corresponding equation for our definite integral J_n . It will be:

$$\int_0^1 x^n e^{2x} dx = \left[\frac{1}{2} x^n e^{2x} \right]_0^1 - \frac{n}{2} \int_0^1 x^{n-1} e^{2x} dx$$

Evaluating the first term on the right-hand side gives us:

$$\left[\frac{1}{2} x^n e^{2x} \right]_0^1 = \frac{1}{2} \cdot 1 \cdot e^2 - 0 = \frac{e^2}{2}$$

Using this, we can abbreviate the equation for J_n using the subscripted- J notation in the following way:

$$J_n = \frac{e^2}{2} - \frac{n}{2} J_{n-1}$$

which is our reduction formula for the definite integral we have chosen. (We might note in passing that our choice of zero as the lower limit of integration makes everything just that bit simpler!)

To round off this example, let's apply this reduction formula to the evaluation of a our definite integral for a particular value of n . In [Section 5 Example 2](#) above, we integrated $x^3 e^{2x}$ using repeated integration by parts. Subsequently, in [Section 6](#), we were able to abbreviate the process a little by using the tabular approach to this repeated process. Here, noting that this integral is just I_3 , to use our current notation, we will proceed to evaluate this

integral with the aid of our reduction formula. We will then proceed to evaluate the corresponding definite integral J_3 in a similar way.

First, then, let's look at the indefinite integral for the case $n = 3$. This is as follows:

$$I_3 = \int x^3 e^{2x} dx$$

With the help of our reduction formula, we can say:

$$I_3 = \frac{1}{2}x^3 e^{2x} - \frac{3}{2}I_2$$

Substituting the expression which the reduction formula gives for I_2 :

$$I_3 = \frac{1}{2}x^3 e^{2x} - \frac{3}{2}\left(\frac{1}{2}x^2 e^{2x} - I_1\right) = \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{2}I_1$$

We can use the reduction formula one more time to obtain an expression for I_1 , which can be substituted into the above expression to obtain:

$$I_3 = \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{2}\left(\frac{1}{2}x e^{2x} - \frac{1}{2}I_0\right)$$

At this point, we need to establish what we obtain when we evaluate I_0 . This is fairly straightforward, and is as follows:

$$I_0 = \int e^{2x} dx = \frac{1}{2}e^{2x} + C$$

(The factor in x is not stated explicitly at this point, because by the laws of indices, $x^0 = 1$.)

Substituting this into our last expression for I_3 and multiplying out the brackets, we are left with a final solution of:

$$I_3 = \int x^3 e^{2x} dx = \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{4}x e^{2x} - \frac{3}{8}e^{2x} + C$$

Reference back to the examples concerned with this integral in [Sections 5 and 6](#) above will confirm that this is the same solution as the ones we obtained in those two previous encounters with this integral.

Finally, we now move on to the definite integral:

$$J_3 = \int_0^1 x^3 e^{2x} dx$$

We will once again apply a reduction formula – this time, the one we obtained for the definite integral J_n . In the case at hand, we can write down:

$$J_3 = \frac{e^2}{2} - \frac{3}{2}J_2$$

We proceed as we did before with the indefinite integral. Substituting in the reduction formula for J_2 gives:

$$J_3 = \frac{e^2}{2} - \frac{3}{2} \left(\frac{e^2}{2} - \frac{2}{2}J_1 \right) = \frac{e^2}{2} - \frac{3e^2}{4} + \frac{3}{2}J_1$$

noting in the above chain of equalities that $2/2 = 1$. Once more, we substitute in the reduction formula for J_1 to obtain:

$$J_3 = \frac{e^2}{2} - \frac{3e^2}{4} + \frac{3}{2} \left(\frac{e^2}{2} - \frac{1}{2}J_0 \right) = \frac{e^2}{2} - \frac{3e^2}{4} + \frac{3e^2}{4} - \frac{3}{4}J_0 = \frac{e^2}{2} - \frac{3}{4}J_0$$

At this point, we must actually evaluate J_0 by hand. This will produce the following result:

$$J_0 = \int_0^1 e^{2x} dx = \left[\frac{1}{2} e^{2x} \right]_0^1 = \frac{1}{2} e^2 - \frac{1}{2}$$

Substituting this back into our last expression for J_3 will give us our final solution of:

$$J_3 = \int_0^1 x^3 e^{2x} dx = \frac{1}{2} e^2 - \frac{3}{8} e^2 + \frac{3}{8} = \frac{e^2}{8} + \frac{3}{8} = \frac{e^2 + 3}{8}$$

We have now integrated $x^3 e^{2x}$ in three different ways: first, using repeated integration by parts; second, abbreviating this by the tabular approach; and now, finally, by creating and using a reduction formula for the integration of $x^n e^{2x}$ and applying it to the case $n = 3$. We note with our customary relief that the solutions given by each of the three techniques are identical.

Example 3

We will now look at a reduction formula example involving a function from the opposite end of the **LIATE** list of function types – a logarithmic function.

Our integral will be the following:

$$I_n = \int \ln^n x \, dx$$

and our efforts to derive a reduction formula for this integral will produce some interesting results, as we shall see. (I will be concerned here solely with the derivation of a reduction formula: those who would like to see the construction and proof of an explicit formula for the integral of $\ln^n x$ based on the reduction formula we produce here may consult [Appendix 5 Section 14.](#))

Integration by Parts

A First Attempt

Remembering the approach we took to the trigonometric integrals earlier in this section, we might consider starting by re-expressing the integrand in this way:

$$\ln^n x = \ln x \cdot \ln^{n-1} x$$

and then choosing parts as follows:

$$u = \ln^{n-1} x \text{ and } \frac{dv}{dx} = \ln x$$

justifying this choice by noting that $\ln x$ can be easily integrated, (but only because we did the job using a clever “trick” in [Section 8](#)) and $\ln^{n-1} x$ is not difficult to differentiate. We obtain the following initial consequences from the suggested choices:

$$\frac{du}{dx} = \frac{(n-1)\ln^{n-2} x}{x} \text{ and } v = x \ln x - x$$

This would give us the following right-hand side integrand from the integration by parts formula:

$$\frac{du}{dx} v = \frac{(n-1)\ln^{n-2} x \cdot x \ln x}{x} - \frac{(n-1)\ln^{n-2} x \cdot x}{x}$$

This would reduce by cancellation of the x terms in each rational expression to the following:

$$(n-1)\ln^{n-1} x - (n-1)\ln^{n-2} x$$

All this means that the integration by parts formula will provide the following equivalence:

$$\int \ln^n x \, dx = \ln^{n-1} x (x \ln x - x) - \int ((n-1)\ln^{n-1} x - (n-1)\ln^{n-2} x) dx$$

The right-hand side can be rewritten, after some multiplying-out and the taking of the coefficients $(n-1)$ outside the integral sign, as follows:

$$x \ln^n x - x \ln^{n-1} x - (n-1) \int \ln^{n-1} x dx + (n-1) \int \ln^{n-2} x dx$$

Rewriting this using the subscripted- I notation which we defined at the outset, we have the following reduction formula:

$$I_n = x \ln^n x - x \ln^{n-1} x - (n-1)I_{n-1} + (n-1)I_{n-2}$$

This is unusual in relation to the other reduction formulae we have derived in this section, since it refers on its right-hand side to *two* other integrals with lower indices, namely both I_{n-1} and I_{n-2} . It is, of course, possible that such a dependency does genuinely exist – certainly there would seem nothing wrong with the approach we took to the work of

integration. However, let's see if a different approach might actually give us a different, even simpler result.

A Simpler Result

This time, we will try to integrate the general integral I_n by taking the approach we took in [Section 8](#) when we integrated the function $\ln x$ (which is, after all, simply I_1 using our current notation). In line with that approach, this time we will re-express the integrand as follows:

$$\ln^n x = 1 \cdot \ln^n x$$

We can now apply the **LIATE** rule to our choice of parts, which will be these:

$$u = \ln^n x \text{ and } \frac{dv}{dx} = 1$$

These choices give us the following consequences:

$$\frac{du}{dx} = \frac{n \ln^{n-1} x}{x} \text{ and } v = x$$

and, taking all these into account, the integration by parts formula gives us:

$$I_n = x \ln^n x - \int n \ln^{n-1} x \, dx$$

(In the right-hand side integral, we have cancelled the x term present in the denominator of du/dx with the actual value of v .)

Taking the coefficient n outside the right-hand side integral and then rewriting what remains using our subscripted- I notation,, we immediately obtain the following very simple reduction formula:

$$I_n = x \ln^n x - n I_{n-1}$$

This is a much more attractive reduction formula than the one we produced at our first attempt. It is also easier to use, since it does not require the previous two integrals in the indexed sequence (I_{n-1} and I_{n-2}) to produce a result, but only the one immediately preceding integral I_{n-1} .

Can These Two Results Be Reconciled?

We have seen that two different approaches to the current integral have yielded two reduction formulae which at first sight appear to be quite different. However, I will try to show here that they are not really as different as they seem. In order to do this, we will first remind ourselves of the simpler of the two reduction formulae, which is as follows:

$$I_n = x \ln^n x - n I_{n-1}$$

I will also state the version of this reduction formula which gives us an expression for I_{n-1} ; this version, obtained by simple substitution of n with $n - 1$ throughout the above formula, is as follows:

$$I_{n-1} = x \ln^{n-1} x - (n-1)I_{n-2}$$

With both these formulae before us, we will now examine a kind of combination of these – I set out below the expression we will examine, which is formed by starting with the reduction formula for I_n and adding and subtracting I_{n-1} on the right-hand side. (You will agree, I hope, that adding and subtracting the same thing from an expression leaves its value unaltered.) The result of this operation can be written as follows:

$$I_n = x \ln^n x - nI_{n-1} - I_{n-1} + I_{n-1}$$

This is all very well, but does it get us anywhere? We may start to see the benefits of the approach when we replace the first instance of I_{n-1} (the one immediately following the minus sign) with the right-hand side of the reduction formula for I_{n-1} . This leads to the following equation:

$$I_n = x \ln^n x - nI_{n-1} - (x \ln^{n-1} x - (n-1)I_{n-2}) + I_{n-1}$$

Removing the brackets around the expression we have just substituted and applying the minus sign which precedes it, we obtain:

$$I_n = x \ln^n x - nI_{n-1} - x \ln^{n-1} x + (n-1)I_{n-2} + I_{n-1}$$

The simple step of gathering together the I_{n-1} terms and rearranging both the subscripted- I terms to the end of the right-hand side expression brings us to this final result:

$$I_n = x \ln^n x - x \ln^{n-1} x - (n-1)I_{n-1} + (n-1)I_{n-2}$$

which is nothing more than the earlier, more complicated, reduction formula we obtained. We might therefore feel completely justified in concluding that the two formulae are equally correct in what they tell us about the relationship between I_n and lower-subscripted integrals of the form I_k .

Some Remarks

A few comments on what we have just witnessed might seem in order at this juncture.

My first comment constitutes a question: can we produce alternative reduction formulae for any of the other general integrals we have examined in this section?

Let's look at a case in point:

$$I_n = \int \cos^n x dx$$

At the start of this section, we obtained the following reduction formula for this general integral:

$$I_n = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} I_{n-2}$$

Let's see what happens if we treat this formula in the same way we treated the simpler of the two formulae for our logarithmic function. To begin, we write down the version of the formula corresponding to I_{n-2} . (Note that we choose this rather than I_{n-1} , since the latter does not appear in our reduction formula for the n -th power of the cosine.) This formula is:

$$I_{n-2} = \frac{1}{n-2} \sin x \cos^{n-3} x + \frac{n-3}{n-2} I_{n-4}$$

By analogy with what we did before, we will now add and subtract I_{n-2} on the right-hand side of the reduction formula for I_n , giving:

$$I_n = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} I_{n-2} - I_{n-2} + I_{n-2}$$

Replacing the I_{n-2} immediately following the minus sign (the second-last term) on the right-hand side with the right-hand side of the reduction formula for I_{n-2} gives:

$$I_n = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} I_{n-2} - \frac{1}{n-2} \sin x \cos^{n-3} x - \frac{n-3}{n-2} I_{n-4} + I_{n-2}$$

We now simplify the right-hand side by gathering together the terms in I_{n-2} as follows:

$$\frac{n-1}{n} I_{n-2} + I_{n-2} = \frac{n-1+n}{n} I_{n-2} = \frac{2n-1}{n} I_{n-2}$$

We can now reorder the terms on the right-hand side of our new formula to give the following result:

$$I_n = \frac{1}{n} \sin x \cos^{n-1} x - \frac{1}{n-2} \sin x \cos^{n-3} x + \frac{2n-1}{n} I_{n-2} - \frac{n-3}{n-2} I_{n-4}$$

This is now a reduction formula which shows the dependence of I_n on both I_{n-2} and I_{n-4} .

While we have indeed produced a “new” reduction formula for the integral of $\cos^n x$, there does seem to be a sense in which we have only achieved something rather trivial – given that I_n depends on I_{n-2} plus a trigonometric function, and I_{n-2} depends on I_{n-4} plus another trigonometric function, it is not really all that surprising that I_n can be shown to depend on *both* I_{n-2} and I_{n-4} (plus, of course, some more trigonometric functions). We might indeed expect to be able to repeat the algebraic addition-and-subtraction technique repeatedly to express the dependence of I_n on a whole sequence of integrals of lower index. Furthermore, we might expect to be able to apply this technique to *any* reduction formula which we obtain through integration!

This reasoning should serve to persuade us that the existence of two (apparently) different reduction formulae for $\ln^n x$ is not really as strange as it first seemed. The interesting issue, however, is that we had two different approaches to the integration by parts of $\ln^n x$ at our

disposal, and one of them might be considered as “more efficient” than the other, since it produced a “minimal” reduction formula – where I_n depended solely on I_{n-1} , rather than the longer formula where it was made to depend on two lower-indexed integrals.

This could be seen as a confirmation of the notion that the most efficient way of integrating a function consisting only of a logarithmic function (where we take the product of n logarithmic functions which is $\ln^n x$ to qualify as a logarithmic function) is to re-express it as the logarithmic function multiplied by 1, and then integrate by parts observing the **LIATE** rule in our choice of parts. The other approach which we were tempted into at the start of this example (based on the method which had succeeded with the powers of the sine and cosine) proved less effective in that it produced an unnecessarily complicated reduction formula.

A final observation in this context might be to note that if we were asked to integrate $\ln^2 x$, then rewriting it as $\ln x \cdot \ln x$ would only give us two identical parts from which to make our choices of u and dv/dx , so that it wouldn't really be a very useful decomposition of the integrand. While decomposing higher powers of $\ln x$ in an analogous way would give us a product of the form $\ln x \cdot \ln^k x$, we might well feel that neither of the two parts of this product offers convincing advantages over the other when it comes to choosing parts. In the light of these comments the restatement:

$$\ln^n x = 1 \cdot \ln^n x$$

becomes much more attractive.

We thus see that while there is sometimes more than one way to skin a cat, not all methods are equal in their effectiveness.

Example 4

We will conclude our examination of the derivation and use of reduction formulae with a brief look at an example which might seem considerably more complicated. We will, however, for the sake of simplicity and relevance, confine ourselves to the evaluation of a definite integral case. It is worth spending a moment or two on this further example in order to get some feeling for the range of applications to which we can put the reduction formula approach.

Our example will be the following definite integral:

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

(We have chosen to abbreviate this definite integral as a function B of two variables m and n , rather than use a variant of our earlier subscript notation. This is reasonable, since the value of the definite integral might be expected to vary depending on the values chosen for m and n . The variable x inside the integral is only a “dummy” variable which we could easily choose to replace with another letter, such as t , and we do not expect this variable to be present when we actually evaluate the definite integral to obtain a numeric value. I would also point out that the choice of B to name the function is appropriate since this is one form of a function studied by the mathematician Euler, and known as the *beta function*.)

If we are going to produce a reduction formula for this general integral, then we would do well to get out of the way right at the start the evaluation of the integrals $B(m, 1)$ and $B(1, n)$, since these may prove useful, and also since they are of interest in themselves. Starting with the first of these, we can say:

$$B(m, 1) = \int_0^1 x^{m-1} \cdot 1 dx = \int_0^1 x^{m-1} dx = \left[\frac{1}{m} x^m \right]_0^1 = \frac{1}{m} \cdot 1 - 0 = \frac{1}{m}$$

Similarly:

$$B(1, n) = \int_0^1 1 \cdot (1-x)^{n-1} dx = \int_0^1 (1-x)^{n-1} dx$$

We will deal with this by first considering the indefinite integral:

$$\int (1-x)^{n-1} dx$$

and we will approach it by using the algebraic substitution $u = 1 - x$, so that:

$$\frac{du}{dx} = -1 \text{ and thus } dx = -du$$

This substitution gives us the following integral in terms of u :

$$-\int u^{n-1} du = -\frac{1}{n} u^n + C$$

Reversing the substitution brings us to the following solution in terms of x :

$$-\frac{1}{n} (1-x)^n + C$$

Armed with this solution, we can now return to our definite integral with the following result:

$$B(1, n) = \int_0^1 (1-x)^{n-1} dx = \left[-\frac{1}{n} (1-x)^n \right]_0^1 = 0 - \left(-\frac{1}{n} \right) = \frac{1}{n}$$

So far, so good. Let's now see what we can make of the task of integrating the general integral $B(m, n)$. Recalling that:

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

We might try proceeding by parts. (We may remember integrating something vaguely similar in [Section 4 Example 3](#) above.) In order to choose parts (first remarking as an aside that the **LIATE** rule cannot help us with the product of two functions of the same type), we

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might note that x^{m-1} is easy to integrate, while the other factor in our product, $(1-x)^{n-1}$ is less easy to integrate (see the approach by substitution a few lines above) – it can, however, be differentiated easily using the Chain Rule.

We will therefore make the following choice:

$$u = (1-x)^{n-1} \text{ and } \frac{dv}{dx} = x^{m-1}$$

From these choices, we can deduce that:

$$\frac{du}{dx} = -(n-1)(1-x)^{n-2} \text{ and } v = \frac{1}{m}x^m$$

The integration by parts formula (bearing in mind that we are dealing here with a *definite* integral) now gives us:

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx = \left[\frac{1}{m}x^m(1-x)^{n-1} \right]_0^1 + \frac{n-1}{m} \int_0^1 x^m(1-x)^{n-2}dx$$

(Note here that the expected minus sign on the right-hand side has been changed to a plus sign because of the minus sign present in du/dx . I have also taken the combined constant coefficient in the right-hand side integral outside the integral sign.)

Evaluating the first term on the right-hand side, we find that:

$$\left[\frac{1}{m}x^m(1-x)^{n-1} \right]_0^1 = 0 - 0 = 0$$

so that, rewriting the right-hand side integral using the functional notation we have adopted, we arrive at the following:

$$B(m, n) = \frac{n-1}{m} B(m+1, n-1)$$

If the right-hand side of this is not immediately clear, it can be understood more easily by noting that:

$$B(m+1, n-1) = \int_0^1 x^{m+1-1}(1-x)^{n-1-1}dx = \int_0^1 x^m(1-x)^{n-2}dx$$

This is a surprisingly simple reduction formula, although it contains one interesting feature: $B(m, n)$ is found to depend on a multiple of $B(p, q)$ where q is less than n , fully in line with the reduction formulae we have derived elsewhere in this section; however, the other variable, p , is actually *greater than* m ; this is the first time we have come across a reduction formula which contains a larger parameter on the right-hand side than on the left.

Integration by Parts

I have taken the consideration of the function $B(m, n)$ a little further in [Appendix 5 Section 15](#), where I set out a brief investigation which leads to the derivation of an explicit integral-free formula based on m and n alone for the value of this function.

Part 3

Algebraic and Trigonometric Methods

In this part of the book, I propose to illustrate and explain the use of a number of algebraic and trigonometric approaches which will help us to integrate some types of function which we have not yet examined. These techniques generally involve carrying out some manipulation on the function we wish to integrate before we actually attempt the integration itself – our aim will be to rearrange or restate our integrand in such a way that we can use techniques of integration which we already know to deal fairly straightforwardly with the restated integrand.

I will also present a number of integration examples which call for the successive use of more than one technique. In the last section of [Part 1](#), we encountered some integrands which required the repeated application of the substitution technique, in one or more of its forms; and [Part 2 Section 5](#) introduced some functions which called for repeated integration by parts; so it should not come as a complete surprise that sometimes just one technique (or one application of it) may not be enough! You will also discover that the application of an algebraic or trigonometric manipulation to an integrand often produces a transformed integrand which needs one or more of the techniques we have already illustrated before the process of integration can be completed.

1 Partial Fractions

In this section, we will look at an algebraic approach based on a technique for restating certain rational functions as the sum or difference of simpler expressions. The “rational functions” I refer to here are functions of x which are expressed in the form:

$$\frac{f(x)}{g(x)}$$

Here, both $f(x)$ and $g(x)$ are themselves functions of x - generally polynomials.

Examples of the kind of rational function we will learn to deal with here include the following:

$$\frac{2}{(x+2)(x-3)}$$

$$\frac{7x+5}{(x-1)(x+2)}$$

The method I will demonstrate here was taught routinely in the past as part of a student’s early acquaintanceship with algebra. However, many students do not have familiarity with

the method nowadays, so I will try to provide careful explanation as I illustrate its use by means of a number of examples.

Example 1

Our first example to illustrate this useful algebraic method will require us to evaluate the following integral:

$$I = \int \frac{6}{(x-4)(x-1)} dx$$

We have never encountered an integrand quite like this in the book so far, and it looks as if none of our techniques can immediately be brought to bear. However, it may be possible to restate the integrand in a form which will let us integrate it using methods we already know. In particular, if we were able to restate the integrand as a sum or difference of simpler rational functions, then we might be able to deal with it term by term in ways with which we are familiar.

To start with, then, we will ask: is it possible to re-express this integrand in the following way:

$$\frac{6}{(x-4)(x-1)} = \frac{A}{x-4} + \frac{B}{x-1}$$

If this were possible, then we could more easily integrate each of the two terms on the right-hand side.

In order to find out whether such a restatement is available to us, let's start with the right-hand side of the above equation and see what we can say about the numerators A and B if we add the two fractions together. In order to perform this addition, we will need to express the right-hand side as one single fraction with a denominator which is a multiple (preferably the lowest common multiple) of $x-4$ and $x-1$. One appropriate denominator for the combined fraction would, of course, be the denominator we already have on the left-hand side, namely $(x-4)(x-1)$. Rewriting the right-hand side as one fraction with this denominator would give us:

$$\frac{A(x-1)}{(x-4)(x-1)} + \frac{B(x-4)}{(x-4)(x-1)} = \frac{A(x-1) + B(x-4)}{(x-4)(x-1)}$$

We can multiply out the brackets in the numerator of the right-hand side expression as follows:

$$\frac{Ax - A + Bx - 4B}{(x-4)(x-1)} = \frac{(A+B)x - (A+4B)}{(x-4)(x-1)}$$

(In the right-hand side above, I have simply gathered like terms together to make the next step easier.)

We now just have to ask ourselves: are there values of A and B which satisfy the following equation – obtained by equating the numerator on the right-hand side above with the numerator of our original integrand):

$$(A + B)x - (A + 4B) = 6$$

or alternatively:

$$(A + B)x - (A + 4B) = 0.x + 6$$

I have expressed the right-hand side the way I have to bring out the fact that since there is no term in x in the numerator of the original integrand, this is equivalent to that numerator being an expression where the coefficient of x is zero. By equating coefficients of like terms, this will actually give us two simultaneous equations in A and B , which are as follows:

$$A + B = 0 \text{ and } -(A + 4B) = 6$$

We can restate these equations as follows, in order to remove the opening minus sign in the second equation:

$$A + B = 0 \text{ and } A + 4B = -6$$

(In this version of the second equation, I have simply moved the minus sign to the right-hand side – in other words, I have multiplied both sides by -1 .)

The first equation can be rearranged to give:

$$B = -A$$

and substituting this into the second equation gives us the following expression for A :

$$A - 4A = -3A = -6 \text{ so that } A = 2$$

The immediately preceding equation for B gives us very simply:

$$B = -2$$

At this stage, we can remind ourselves that this information arose from our attempt to restate our integrand in this example as follows:

$$\frac{6}{(x-4)(x-1)} = \frac{A}{x-4} + \frac{B}{x-1}$$

The conclusions we have just reached about the values of A and B which would make this true now let us write the following equivalence:

$$\frac{6}{(x-4)(x-1)} = \frac{2}{x-4} - \frac{2}{x-1}$$

In case this conclusion seems too good to be true, let's take a moment to carry out the addition of the two fractions on the right-hand side with their actual numerators in place, rather as we did with the symbolic numerators A and B . We get the following:

$$\frac{2(x-1) - 2(x-4)}{(x-4)(x-1)} = \frac{2x-2-2x+8}{(x-4)(x-1)}$$

The numerator on the right-hand side reduces to 6 (the $2x$ terms cancel each other out), and this value matches the numerator in our original integrand – so we can feel quite confident that our restated integrand is indeed correct.

All that remains is to integrate the resulting two-term expression. We first note that our original integral I can now be rewritten as follows:

$$I = \int \frac{6}{(x-4)(x-1)} dx = 2 \int \frac{1}{x-4} dx - 2 \int \frac{1}{x-1} dx$$

I have, of course, taken the coefficients of 2 which featured in each term in the two-term expansion of our integrand outside their respective integral signs. Having done this, it should be clear that we can now apply one of the integration rules we derived using substitution in [Part 1 Section 7](#) – in particular the rule which states that:

$$\int \frac{1}{x+k} dx = \ln|x+k| + C$$

Applying this rule to the two-term expansion of our integral I , we have the following solution:

$$I = 2 \ln|x-4| - 2 \ln|x-1| + C = 2(\ln|x-4| - \ln|x-1|) + C$$

Well, this is obviously a solution – one we would not have been able to write down without the use of the algebraic method explained above. However, we can still do a little better. Recalling the rule of logarithms which states that:

$$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$$

We can finally restate the solution as follows:

$$\int \frac{6}{(x-4)(x-1)} dx = 2 \ln \left| \frac{x-4}{x-1} \right| + C$$

Some Comments

Let's take a moment to review the steps we took in evaluating the above integral. We started with an integrand whose denominator consisted of the product of two linear factors. We immediately threw in the towel, and decided that the only way to tackle this was to restate it, if possible, as the sum of two rational expressions, each of whose denominators was a single linear factor. These rational expressions could be integrated easily using a standard rule we derived in [Part 1 Section 7](#).

We set out our restated version of the integrand, using the letters A and B as placeholders for the numerators of the two terms of the restated expression, and then tried to find out what we could say about A and B once we had done the algebra necessary to add the two terms together to create one single rational expression with the same denominator as our original integrand. This gave us an expression involving both A and B , which we could then equate to our original numerator of 6.

The expression involving A and B could be rearranged to take the form $Px + Q$, which we could set equal to 6. We then equated coefficients of like terms, which led to two equations which had the form:

$$P = 0 \text{ and } Q = 6$$

since the numerator of the original integrand contained no x terms.

In this summary, we should remember that P and Q in the above abbreviated description both stand for expressions involving A and B , so that the process of equating coefficients of like terms actually gave us two simultaneous equations in A and B . These were then solved, so that we were finally able to express our integrand in the form:

$$\frac{A}{x-4} + \frac{B}{x-1}$$

which could be integrated term by term quite easily using one of our standard rules.

Finally, the solution to our original integral turned out to contain the difference of two logarithmic expressions, which we were able to combine into a neater form using the following rule of logarithms:

$$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$$

The algebraic technique we used to decompose the original integrand into two separate terms is known as the *Method of Partial Fractions*. We will use the same broad approach to handle the remaining examples in this section, although each of these will be chosen so as to bring out some particularly useful point about the method.

An Alternative Way to Determine A and B

In the above example, we determined the necessary values of A and B by creating two simultaneous equations in the two variables and then solving those equations. There is an alternative way to arrive at the values of these placeholders, which may seem simpler, particularly to those less comfortable with solving simultaneous equations.

In order to explain this alternative method, let's cast our minds back to the equality we set up between our original integrand and its expression as a sum of two separate rational expressions. The equivalence was as follows:

$$\frac{6}{(x-4)(x-1)} = \frac{A}{x-4} + \frac{B}{x-1} = \frac{A(x-1) + B(x-4)}{(x-4)(x-1)}$$

By equating numerators in the first and last of these expressions, we have the following equation, which must remain true whatever the value taken by the variable x :

$$A(x-1) + B(x-4) = 6$$

Since this must be true whatever value is taken by x , we can experiment by setting x equal to a number of different values and investigating what this tells is about A and B .

To start with, let's choose a value for x which makes the factor $x-4$ which is multiplied by B equal to zero. This value is easy to choose – it will be $x = 4$. With this value for x , our equation will become:

$$3A = 6 \text{ so that } A = 2$$

Let's now try the analogous step which will make the factor $x-1$ which is multiplied by A equal to zero. This, of course, requires that we set $x = 1$. Our equation then becomes:

$$-3B = 6 \text{ so that } B = -2$$

As a final experiment, let's see what happens if we choose $x = 2$. Our equation then becomes:

$$A - 2B = 6$$

which is indeed satisfied by the values of A and B which we obtained a little earlier. We can also confirm (as we might have hoped) that these values of A and B are identical to those obtained by the simultaneous equation approach we saw in the earlier work on our example.

It might be argued that this alternative approach is a good deal easier than setting up and solving simultaneous equations, and I will use this as my preferred method in the examples which follow.

Example 2

In the work we did to rearrange the integrand in [Example 1](#) into a form we could integrate easily, we obtained an equation relating parts of the rearranged integrand with the numerator of the original integrand. This equation was as follows:

$$A(x-1) + B(x-4) = 6$$

Looking again at this equation, we might suspect that the same approach we took to [Example 1](#) might work for rational functions with linear expressions in their numerator, rather than simply a constant. In other words, we might apply it to the integration of a function of an expression of the following form:

$$\frac{px + q}{(x + a)(x + b)}$$

Exploring this suspicion a little further, we will now try to apply the Method of Partial Fractions to the following integral:

$$I = \int \frac{7x + 5}{(x - 1)(x + 2)} dx$$

This is, of course, the second example of a rational function I gave at the beginning of this section.

As in [Example 1](#), we will aim to restate the integrand in the following form:

$$\frac{7x + 5}{(x - 1)(x + 2)} = \frac{A}{x - 1} + \frac{B}{x + 2}$$

and set about determining the values of A and B which will make this equation true.

As before, we will rewrite the right-hand side using the denominator $(x - 1)(x + 2)$ - this is a common denominator for the two denominators $x - 1$ and $x + 2$ on the right-hand side - and we will arrive at the following expression:

$$\frac{A(x + 2)}{(x - 1)(x + 2)} + \frac{B(x - 1)}{(x - 1)(x + 2)} = \frac{A(x + 2) + B(x - 1)}{(x - 1)(x + 2)}$$

We can now equate the numerator on the right-hand side above with the numerator of our original integrand to obtain the following equation:

$$A(x + 2) + B(x - 1) = 7x + 5$$

Rather than multiplying out the left-hand side and equating coefficients so as to derive two simultaneous equations in A and B , we will on this occasion take the rather quicker route of realising that since the above equation must hold for all values of x , we can proceed by choosing suitable values to produce a single equation for each of A and B .

First, in order to make the B term in the above equation disappear, we must choose $x = 1$, which will make its factor of $x - 1$ equal to zero. This choice gives us:

$$3A = 12 \text{ so that } A = 4$$

For the A term to vanish, we must choose $x = -2$, producing the following equation for B :

$$-3B = -9 \text{ so that } B = 3$$

These values for A and B allow us to rewrite our original integrand in the desired form as follows:

$$\frac{7x + 5}{(x - 1)(x + 2)} = \frac{4}{x - 1} + \frac{3}{x + 2}$$

To verify that we have not been inaccurate – and indeed that our approach is a valid one – let's perform the addition of the two fractional expressions on the right-hand side of the above equation, as follows:

$$\frac{4(x + 2) + 3(x - 1)}{(x - 1)(x + 2)} = \frac{4x + 8 + 3x - 3}{(x - 1)(x + 2)}$$

We can see that the numerator on the right-hand side reduces to $7x + 5$ when like terms are added together.

Now we can move to the actual evaluation of our integral. The Method of Partial Fractions has allowed us to rewrite the integrand, so that we can now say that:

$$I = \int \frac{7x + 5}{(x - 1)(x + 2)} dx = 4 \int \frac{1}{x - 1} dx + 3 \int \frac{1}{x + 2} dx$$

I have, of course, taken the numerators of the two separate fractional expressions outside the integral signs on the right-hand side, treating them as constant coefficients.

The right-hand side now integrates easily using one of our standard rules for integration to give us the following result:

$$I = 4 \ln|x - 1| + 3 \ln|x + 2| + C$$

Unlike the result in the previous example, we don't have any common factors in this solution, so that we can go no further in simplifying it, and must be content with what we have obtained.

Example 3

In this example, we will take a couple of steps on from our two opening examples, and consider an integral which at first sight might not seem to be suitable for the Method of Partial Fractions. However, appearances can (sometimes) be deceptive, as we shall see.

Our integral will be the following one:

$$I = \int \frac{2x + 3}{x^2 - 9} dx$$

Clearly, the denominator here does not conform to the pattern established by the previous two examples. However, we should recall a standard but sometimes easily overlooked rule of algebra, which states the following identity:

$$a^2 - b^2 = (a + b)(a - b)$$

Noting that the second term in the denominator, 9, is a perfect square, we can restate the denominator using the above rule as follows:

$$x^2 - 9 = (x + 3)(x - 3)$$

Bear in mind that, even if the second term in the denominator were not a perfect square – perhaps 7, for example, instead of 9 – we would still be able to restate the denominator using a square root, as follows:

$$x^2 - 7 = (x + \sqrt{7})(x - \sqrt{7})$$

But we must return to the integral in hand. Having restated the denominator of the integrand as a product of two linear factors, we find our integral taking on a much more familiar shape:

$$I = \int \frac{2x + 3}{(x + 3)(x - 3)} dx$$

We can handle this in exactly the same way we handled [Example 2](#) above. However, I did indicate that the current example would take us forward *two* steps, so do keep your eyes open for a second new feature when we actually get our teeth into the algebra required.

We now want to restate our integrand in the following form:

$$\frac{2x + 3}{(x + 3)(x - 3)} = \frac{A}{x + 3} + \frac{B}{x - 3}$$

And we will move towards this restatement by rewriting the right-hand side above using the common denominator $(x + 3)(x - 3)$ as follows:

$$\frac{A(x - 3)}{(x + 3)(x - 3)} + \frac{B(x + 3)}{(x + 3)(x - 3)} = \frac{A(x - 3) + B(x + 3)}{(x + 3)(x - 3)}$$

Equating the numerator on the right-hand side with the numerator of our original integrand, we have this equation, true for all values of x :

$$A(x - 3) + B(x + 3) = 2x + 3$$

We will now select particular values of x which will make first B and then A disappear from this equation, in order to find values for these two placeholders.

First, setting $x = -3$ will zero the factor $x + 3$, and give us the following:

$$-6A = -3 \text{ so that } A = \frac{1}{2}$$

Now, setting $x = 3$ will zero the factor $x - 3$, so that we will have:

$$6B = 9 \text{ so that } B = \frac{3}{2}$$

These conclusions should immediately give us pause, since we have not so far encountered a situation where the values of A and B were found to be *fractions*. These values will give us the following new version of our integrand:

$$\frac{2x + 3}{(x + 3)(x - 3)} = \frac{1/2}{x + 3} + \frac{3/2}{x - 3}$$

This rather ugly right-hand side is more commonly written as follows:

$$\frac{1}{2(x + 3)} + \frac{3}{2(x - 3)}$$

and we might consider this a form which is not so easily integrated as the partial fraction representations in the first two examples. However, all is not lost if we look back briefly at the “ugly” right-hand side I presented a few lines ago, and see that this representation of our integrand can be written as the sum of two rational expressions, each of which is multiplied by a fractional coefficient, as follows:

$$\frac{1}{2} \cdot \frac{1}{x + 3} + \frac{3}{2} \cdot \frac{1}{x - 3}$$

This expression then becomes easy to integrate, moving the coefficients outside the integral sign for each term, as follows:

$$I = \int \frac{2x + 3}{(x + 3)(x - 3)} dx = \frac{1}{2} \int \frac{1}{x + 3} dx + \frac{3}{2} \int \frac{1}{x - 3} dx$$

The right-hand side here will integrate using the same rule of integration we have been relying upon throughout this section, to give the following solution:

$$I = \int \frac{2x + 3}{x^2 - 9} dx = \frac{1}{2} \ln|x + 3| + \frac{3}{2} \ln|x - 3| + C$$

The best we can do in simplifying this solution further would be to factorise out $1/2$, but this doesn't offer us much more elegance, so I won't go to the trouble.

It's worth concluding this example, however, with quick reference to the two distinctive aspects of the integrand we have been dealing with. First, the denominator was not presented in the usual form for the Method of Partial Fractions, but we were able to transform it into that form by using a standard rule of algebra; and second, the algebra we carried out gave rise to fractional values for the placeholders A and B - however, we saw that each of these could simply be treated as a coefficient of its corresponding rational expression in x , and subsequently taken outside the integral sign, thus proving no real trouble at all.

Two Standard Integrals

Before we delve into some more advanced features of the algebraic approach we are dealing with here, this is an opportune moment to mention two integrals with symbolic constant terms which are often quoted and used without much justification. Both of these rules for integration can be proved using the Method of Partial Fractions, and I have set out proofs for

both of them in [Appendix 5 Section 16](#) - the greater part of the purely symbolic reasoning in this book has been relegated to [Appendix 5](#).

These rules, which are also listed for reference in [Appendix 4 Section 2](#), are as follows:

$$\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{a-b} \ln \left| \frac{x+b}{x+a} \right| + C$$

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

Although the full detail of the evaluation of the integrals in the above rules can be found in [Appendix 5](#), it will be instructive to check the solutions predicted by these rules with any of the examples we have attempted earlier in this section to which we can apply the rules.

Let's take [Example 1](#) as a case in point – here we evaluated our chosen integral as follows:

$$\int \frac{6}{(x-4)(x-1)} dx = 2 \ln \left| \frac{x-4}{x-1} \right| + C$$

We can apply the first rule of integration set out a few lines ago to this integral if we are prepared to restate it as follows:

$$\int \frac{6}{(x-4)(x-1)} dx = 6 \int \frac{1}{(x-4)(x-1)} dx$$

If we now set $a = -1$ and $b = -4$, noting as a consequence that on the right-hand side of our rule of integration this means that:

$$a - b = -1 - (-4) = -1 + 4 = 3$$

then the rule of integration will give us the following result:

$$\int \frac{6}{(x-4)(x-1)} dx = 6 \cdot \frac{1}{3} \ln \left| \frac{x-4}{x-1} \right| + C = 2 \ln \left| \frac{x-4}{x-1} \right| + C$$

which is exactly what we obtained using the Method of Partial Fractions.

Example 4

In this example, we will take our technique a little further, by considering the following integral:

$$I = \int \frac{x^2 + x - 1}{x(x^2 - 1)} dx$$

The integrand here looks somewhat different from those we have so far dealt with in two ways: first, the numerator is a *quadratic* expression, instead of the linear expressions we have

been looking at so far; and second, the denominator does not at first seem to comprise a product of linear expressions in x .

The second of these issues can be fairly easily neutralised: first, the factor x can, of course, be viewed as the linear expression $x + 0$; and second, the remaining quadratic factor can be rewritten using simple algebra in the following way:

$$x^2 - 1 = (x + 1)(x - 1)$$

(This is the kind of rearrangement we made use of in order to express the denominator of the integrand in [Example 3](#) above as a product of linear factors.)

Our integral now looks a little less problematic, since it can be expressed as follows:

$$\int \frac{x^2 + x - 1}{x(x + 1)(x - 1)} dx$$

The numerator of the integrand is, of course, still a quadratic; however, perhaps this will not cause us too much extra difficulty, since we might also note that the denominator will multiply out to produce a cubic expression – this means that the order of the polynomial in the denominator will still be greater than the order of the polynomial in the numerator, and this was the case also in our previous three examples.

So let's proceed with a similar approach to the one we have taken before, and see how far we can get. We will try to restate our integrand in the following form – allowing one term for each linear factor in the denominator:

$$\frac{A}{x} + \frac{B}{x + 1} + \frac{C}{x - 1}$$

which will allow us to integrate easily term by term.

In order to establish values for A , B and C . We will as before restate the above sum of three rational expressions as a single rational expression with the common denominator of $x(x + 1)(x - 1)$, as follows:

$$\frac{A(x + 1)(x - 1) + Bx(x - 1) + Cx(x + 1)}{x(x + 1)(x - 1)}$$

We can now equate the numerator of this expression with the numerator of our original integrand to obtain the following equation:

$$A(x + 1)(x - 1) + Bx(x - 1) + Cx(x + 1) = x^2 + x - 1$$

and we now go on to choose values for x which will allow us to obtain simple equations for each of the three values we are seeking.

First, choosing $x = 0$ will successfully remove B and C from the picture, giving us the following:

$$-A = -1 \text{ so that } A = 1$$

If we now choose $x = -1$, we will obtain the following equation for B :

$$2B = -1 \text{ so that } B = -\frac{1}{2}$$

Finally, selecting $x = 1$ will give us the following equation for C :

$$2C = 1 \text{ so that } C = \frac{1}{2}$$

We can now substitute these values into our version of the integrand containing three rational expressions to end with the following equivalence:

$$\frac{x^2 + x - 1}{x(x+1)(x-1)} = \frac{1}{x} - \frac{1}{2} \cdot \frac{1}{x+1} + \frac{1}{2} \cdot \frac{1}{x-1}$$

The right-hand side can be integrated easily term by term, as follows:

$$\int \frac{1}{x} dx - \frac{1}{2} \int \frac{1}{x+1} dx + \frac{1}{2} \int \frac{1}{x-1} dx = \ln|x| - \frac{1}{2} \ln|x+1| + \frac{1}{2} \ln|x-1| + C$$

As a final tidy-up, we can reverse the order of the two latter logarithmic functions and factorise out the common coefficient of $1/2$, to arrive at the following solution:

$$I = \ln|x| + \frac{1}{2} (\ln|x-1| - \ln|x+1|) + C = \ln|x| + \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + C$$

The final step here (and, indeed the reason why I wanted to reverse the order of the logarithmic functions) follows from an application of the rule of logarithms which we have used several times already in this section.

Some Remarks

We can see in this example that the technique we applied in the simpler cases where our integrand was the product of two linear factors can also be applied successfully where three linear factors are present. It is worth asking, however, whether the integrand in this example was just a lucky choice, which allowed us to obtain our values for A , B and C quite easily. Indeed, we might ask: would it always be possible to choose values for x which produced the necessary simple equations, whatever the linear factors in the denominator.

We can go some way towards answering this question by looking at the general example of an expression of the form:

$$\frac{Q(x)}{(x+a)(x+b)(x+c)}$$

where $Q(x)$ is some quadratic in x . If we want to express this in the expanded form:

$$\frac{A}{x+a} + \frac{B}{x+b} + \frac{C}{x+c}$$

Then we will need to find a numerator in terms of A , B and C using the common denominator $(x+a)(x+b)(x+c)$. With this denominator, our expanded expression is equivalent to:

$$\frac{A(x+b)(x+c) + B(x+a)(x+c) + C(x+a)(x+b)}{(x+a)(x+b)(x+c)}$$

We can see from examining the numerator here that each of the linear factors appears exactly twice; indeed each of the linear factors appears as a factor of exactly two out of the three placeholders A , B and C . This means that we can always choose values for x which will produce simple equations for each of these placeholders as follows:

$$x = -a \text{ produces the equation } A(b-a)(c-a) = Q(-a)$$

$$x = -b \text{ produces the equation } B(a-b)(c-b) = Q(-b)$$

$$x = -c \text{ produces the equation } C(a-c)(b-c) = Q(-c)$$

Each of these reduces to a simple equation of the form:

$$kX = Q(n)$$

where X stands for one of the placeholders A , B or C and n represents the value we have chosen for x .

This should be enough to convince us that the Method of Partial Fractions as we have deployed it where our denominators have comprised just two linear factors will also be expected to work where there are three such factors.

2 Partial Fractions with Non-Linear Factors in the Denominator

There are two more types of rational expression where a variant of the Method of Partial Fractions can be used. The first of these, which we will look at in the example below, will contain a *squared* linear factor in its denominator. The second type will contain an *irreducible quadratic factor* in its denominator, which will require a refinement of our approach.

Example 1

We will now move one stage further in our exploration of the Method of Partial Fractions as a tool in integration by examining the following integral:

$$I = \int \frac{x+5}{(x-2)(x+1)^2} dx$$

This has an integrand with a *repeated* linear factor in its denominator. In other words, the integrand might be rewritten as follows:

$$\frac{x + 5}{(x - 2)(x + 1)(x + 1)}$$

There is, however, nothing to be gained by attempting to expand it out in the following form, in imitation of the way we expanded the integrand in [Section 1 Example 4](#):

$$\frac{A}{x - 2} + \frac{B}{x + 1} + \frac{C}{x + 1}$$

since this would simply be equivalent to:

$$\frac{A}{x - 2} + \frac{B + C}{x + 1} \text{ which is the same as } \frac{A}{x - 2} + \frac{D}{x + 1} \text{ where } D = B + C$$

This expansion therefore does not constitute the *three* fractional expressions we would expect in the kind of expansion we have been aiming at in previous examples, and the denominator of the second fractional expression here is not the *squared* expression in the original integrand. Instead, we ought to be seeking to express the integrand in the form:

$$\frac{A}{x - 2} + \frac{B}{(x + 1)^2}$$

in line with our earlier examples.

So let's see how far we can get in achieving this objective. As usual, our first step will be to express the expanded version containing the placeholders A and B as one single fraction using a common denominator. This gives us the following:

$$\frac{A(x + 1)^2 + B(x - 2)}{(x - 2)(x + 1)^2}$$

We might at this point try to find values of x which will make the variorum factors multiplying A and B take zero values, so that we can find simple equation for each of the two placeholders. However, it will be easier to demonstrate the point I want to make here by instead choosing the method of equating coefficients between numerators so as to obtain simultaneous equations in A and B .

We can, of course, start by multiplying out the quadratic factor in the numerator as follows:

$$(x + 1)^2 = x^2 + 2x + 1$$

so that our single fraction can be turned into:

$$\frac{A(x^2 + 2x + 1) + B(x - 2)}{(x - 2)(x + 1)^2}$$

The numerator here can be expanded as follows

$$Ax^2 + 2Ax + A + Bx - 2B$$

We can now gather together like terms and equate this expression with our original numerator, as follows:

$$Ax^2 + (2A + B)x + A - 2B = x + 5$$

Equating coefficients between the right-hand side above and the numerator of our original integrand would be expected to give us a set of equations which we can then solve in order to find A and B . However, we have *three* equations in this case, since the numerator we have just obtained also contains a term in x^2 . Our three equations are therefore as follows:

$$\begin{aligned}A &= 0 \\2A + B &= 1 \\A - 2B &= 5\end{aligned}$$

Even before we try to solve these equations, the very first one asserts that $A = 0$ - so that any solution of the form we have been hoping for will not be possible, since it would have only one term because the first one would be equal to zero! This means, of course, that our original model for the solution cannot be valid, because it has led to a contradiction on closer examination. We therefore need another approach.

An Alternative Approach

I will suggest that, instead of the proposed expansion of our integrand which I gave above, that we try an expansion of the form:

$$\frac{A}{x-2} + \frac{Bx+C}{(x+1)^2}$$

Here, we have a linear expression in x as our numerator in the second fraction. The only justification for this choice of format which I will offer at this juncture is to suggest that since the denominator of the second fraction can be multiplied out to produce a quadratic, a linear expression (which is of one degree less than a quadratic) might be more suitable as the numerator of a term with a quadratic denominator. Let's see how far this gets us.

As usual, let's combine the two terms of our proposed expansion into one single fraction with a common denominator. This will give us:

$$\frac{A(x+1)^2 + (Bx+C)(x-2)}{(x-2)(x+1)^2}$$

Expanding out the numerator gives:

$$Ax^2 + 2Ax + A + Bx^2 - 2Bx + Cx - 2C$$

and having gathered together like terms, we can equate this with the original numerator to produce the following equation:

$$(A + B)x^2 + (2A - 2B + C)x + A - 2C = x + 5$$

Equating coefficients then brings us the following three simultaneous equations:

$$\begin{aligned}A + B &= 0 \\2A - 2B + C &= 1 \\A - 2C &= 5\end{aligned}$$

We should pause a moment here to note a further piece of after-the-event justification for trying an expansion with a linear factor over the quadratic part of the denominator: the second part of the numerator we created in order to equate it with the numerator of the original integrand took the form:

$$(Bx + C)(x - 2)$$

When multiplied out, this immediately supplied an additional term in x^2 , so that equating coefficients was able to give us the equation $A + B = 0$, rather than the completely useless $A = 0$ which stymied us when we tried out our first expansion. This is the real reason for the particular alteration to the style of our expansion in the presence of the quadratic factor we had in our original denominator.

Can we now solve the three equations we have obtained, which we were unable to do sensibly with our first expansion? Well, to begin with, the second equation gives us the following expression for C in terms of A and B :

$$C = 1 - 2A + 2B$$

Substituting this expression for C into our third equation gives us the following:

$$A - 2 + 4A - 4B = 5 \text{ so that } 5A - 4B - 2 = 5 \text{ or } 5A - 4B = 7$$

Putting this result next to our first equation will then give us the following two equations in A and B alone:

$$\begin{aligned}A + B &= 0 \\5A - 4B &= 7\end{aligned}$$

The first of these rearranged tells us that:

$$B = -A$$

and when this is substituted into the second of the pair, we have:

$$5A + 4A = 7 \text{ so that } 9A = 7 \text{ and } A = \frac{7}{9}$$

The expression we obtained for B now gives us as a consequence:

$$B = -\frac{7}{9}$$

and finally, the expression we obtained a little earlier for C gives us:

$$C = 1 - 2 \cdot \frac{7}{9} - 2 \cdot \frac{7}{9} \text{ or } C = 1 - \frac{28}{9} \text{ which means that } C = -\frac{19}{9}$$

Substituting these values into our proposed expansion for the original integrand, which we will recall was as follows:

$$\frac{A}{x-2} + \frac{Bx+C}{(x+1)^2}$$

gives the following end result:

$$\frac{x+5}{(x-2)(x+1)^2} = \frac{\left(\frac{7}{9}\right)}{x-2} - \frac{\left(\frac{7}{9}x + \frac{19}{9}\right)}{(x+1)^2} = \frac{1}{9} \cdot \frac{7}{x-2} - \frac{1}{9} \cdot \frac{7x+19}{(x+1)^2}$$

We can factorise out the common coefficient of $1/9$ from this expression to obtain the rather less ugly equivalent:

$$\frac{1}{9} \left(\frac{7}{x-2} - \frac{7x+19}{(x+1)^2} \right)$$

Now even after the factorisation step, this isn't the most attractive expression in the world. We might naturally expect something a little more individual since the single rational expression we began with did not conform to the pattern established in the earlier examples in this section.

However, it is a simple matter to confirm the correctness of our expansion. All we have to do is to evaluate and then simplify the numerator we would obtain if we chose to re-express our expansion as a single fraction using the common denominator $(x-2)(x+1)^2$. (We will do this for the bracketed portion of our solution only, and then apply the coefficient of $1/9$ at the end.)

The numerator we would obtain looks like this:

$$7(x+1)^2 - (7x+19)(x-2)$$

which, on multiplying everything out and taking care to apply the minus sign between our initial two terms as it should be applied, results in:

$$7x^2 + 14x + 7 - 7x^2 + 14x - 19x + 38$$

Once we gather together the like terms in this long expression – noting that the terms in x^2 cancel each other out - we are left with the simple linear expression:

$$9x + 45$$

and once we apply our constant coefficient of $1/9$ to this, we get the numerator we started out with, namely:

$$x + 5$$

So this confirms that our expansion is a valid one. Before proceeding to carry out the integration (which is, after all, the whole point of working out in such detail how to expand the rational expression we began with) we might ask if we can do any better than this. In particular, we should recall that in [Section 1 Example 4](#), we expanded a rational expression with three linear factors in its denominator into a form comprising three separate fractional expressions – one corresponding to each of the three linear terms. In the current example, we also have three linear terms in the denominator – only one of them, the $x + 1$ term, is *repeated*: we could write out the denominator “in full” as $(x - 2)(x + 1)(x + 1)$.

A Three-Term Expansion

Considerations such as these might lead us to think that we ought to be able to produce a three-term expansion for the integrand in this example as well. We have already exhibited some reasoning to the effect that there is no mileage in looking for an expansion containing two fractions with the denominator $x + 1$, since these would in very short order reduce to just one fraction with this denominator – this would be no more than simple fractional arithmetic, rather like saying that:

$$\frac{2}{15} + \frac{4}{15} = \frac{6}{15}$$

In order to see if such a three-term expansion is possible, we will first need to decide on what the best choices for the denominators of our expanded terms ought to be. Once we have decided this, we ought then to go through the procedure of creating a single rational expression from our new expansion and once again equating its numerator with the numerator of our original integrand.

Since we have ruled out multiple fractions with the denominator $x + 1$, but we still seek to create an expansion comprising three fractional terms, probably the only possibility which remains open to us is an expansion in the following format:

$$\frac{A}{x - 2} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}$$

As predicted above, we ought now to restate this expansion as a single rational expression with the common denominator $(x - 2)(x + 1)^2$ so that we can equate its numerator with the numerator of our original integrand. It won't do to reason that we have already obtained a fraction with denominator $x - 2$ when we expanded the integrand into two fractional terms earlier, since the numerator of the fraction with denominator $x - 2$ might well turn out to be quite different in this three-term expansion. So let's proceed to create a single rational expression from our new expansion and carry on from there. Our expression will be:

$$\frac{A(x + 1)^2 + B(x - 2)(x + 1) + C(x - 2)}{(x - 2)(x + 1)^2}$$

(We immediately note that the numerator here has a rather more complicated pattern than those we have seen before.)

Equating numerators as already discussed, we get:

$$A(x+1)^2 + B(x-2)(x+1) + C(x-2) = x+5$$

The left-hand side can be expanded by multiplying out, and gives:

$$A(x^2 + 2x + 1) + B(x^2 - x - 2) + C(x - 2)$$

which in turn expands to give:

$$Ax^2 + 2Ax + A + Bx^2 - Bx - 2B + Cx - 2C$$

We can now gather together like terms to obtain the following final equation of numerators:

$$(A+B)x^2 + (2A-B+C)x + A-2B-2C = x+5$$

Equating coefficients now produces these three simultaneous equations:

$$\begin{aligned}A+B &= 0 \\2A-B+C &= 1 \\A-2B-2C &= 5\end{aligned}$$

Proceeding towards a solution, we can first rearrange the second equation to give the following equation for C :

$$C = 1 - 2A + B$$

Substituting this into the third equation gives:

$$A - 2B - 2 + 4A - 2B = 5 \text{ which reduces to } 5A - 4B = 7$$

We can now combine this with the first of our three equations to get the following two in A and B alone:

$$\begin{aligned}A+B &= 0 \\5A-4B &= 7\end{aligned}$$

The first of these rearranges very simply to give this equation for B :

$$B = -A$$

and we can substitute this into the second of the two equations above to give:

$$9A = 7 \text{ so that } A = \frac{7}{9}$$

The equation for B which we obtained above now gives:

$$B = -\frac{7}{9}$$

and the earlier equation for C now produces:

$$C = 1 - \frac{14}{9} - \frac{7}{9} = -\frac{12}{9} = -\frac{4}{3}$$

Let's now put these values for A , B and C back into our proposed three-term expansion to obtain:

$$\frac{x+5}{(x-2)(x+1)^2} = \frac{7}{9} \cdot \frac{1}{x-2} - \frac{7}{9} \cdot \frac{1}{x+1} - \frac{4}{3} \cdot \frac{1}{(x+1)^2}$$

If we rewrite the final coefficient of $4/3$ as $12/9$, we can factorise out the common factor of $1/9$, allowing us to write down our expansion rather more neatly as:

$$\frac{1}{9} \left(\frac{7}{x-2} - \frac{7}{x+1} - \frac{12}{(x+1)^2} \right)$$

Interestingly, we find that the numerator of the term with denominator $x-2$ *does* in fact turn out to be the same as we found it to be in the case of our earlier two-term expansion.

Just as before, it is probably best to check our working here, by creating the numerator we would get if we re-expressed our three-term expansion as a single rational expression with the common denominator $(x-2)(x+1)^2$ and ensuring that it comes out the same as the numerator in our original integrand. Once again, I propose to leave the common coefficient of $1/9$ out of consideration until the last stage of the comparison, to make the arithmetic simpler.

The numerator our expansion would produce (without the coefficient of $1/9$) is as follows:

$$7(x+1)^2 - 7(x-2)(x+1) - 12(x-2)$$

Multiplying out the bracketed expressions in x turns this into:

$$7(x^2 + 2x + 1) - 7(x^2 - x - 2) - 12(x - 2)$$

and we can now multiply out the brackets to get:

$$7x^2 + 14x + 7 - 7x^2 + 7x + 14 - 12x + 24$$

which, on adding everything up, gives the following result:

$$9x + 45$$

This, of course, when we reapply the factorised-out coefficient of $1/9$, gives the same numerator as our original integrand had, namely:

$$x + 5$$

so we can be confident that our second expansion is also correct.

Carrying Out the Integration

We now turn to what should really qualify as the “main event” for this example, namely the actual integration. All that precedes has merely served to get the integrand into a suitable form for us to carry out this task. Using the three-term expansion we have just derived and confirmed, we can now say that:

$$I = \int \frac{x + 5}{(x - 2)(x + 1)^2} dx = \frac{1}{9} \int \left(\frac{7}{x - 2} - \frac{7}{x + 1} - \frac{12}{(x + 1)^2} \right) dx$$

We will, of course, integrate the right-hand side term by term. In order to do this with as much clarity as is possible, we will divide the right-hand side integral into three separate integrals, and combine the solutions when we have dealt with them all. We will therefore use the following notation:

$$I_1 = \int \frac{7}{x - 2} dx, I_2 = \int \frac{7}{x + 1} dx \text{ and } I_3 = \int \frac{12}{(x + 1)^2} dx$$

With this notation, our final solution will be made up as follows:

$$I = \frac{1}{9} (I_1 - I_2 - I_3)$$

The first two of our three integrals can be written down quickly and easily, using the familiar rule of integration which we have been bringing into play throughout this section. The respective solutions are:

$$I_1 = 7 \ln|x - 2| + C \text{ and } I_2 = 7 \ln|x + 1| + C$$

The third of our component integrals is of a rather different form from anything we have seen in this section so far, but it should not deter us, since we can bring another of the rules of integration we derived using the method of substitution in [Part 1 Section 7](#). The rule we need is the following one:

$$\int (f(x))^n f'(x) dx = \frac{1}{n + 1} (f(x))^{n+1} + C$$

To use this rule, we can rewrite the integral I_3 as follows:

$$I_3 = 12 \int \frac{1}{(x + 1)^2} dx$$

and then apply our rule, with the following specific values for its placeholders:

$$f(x) = x + 1, f'(x) = 1 \text{ and } n = -2$$

Our chosen rule then predicts that:

$$I_3 = 12 \cdot \frac{1}{-1} (x+1)^{-1} + C = -\frac{12}{x+1} + C$$

Applying these three solutions to the formula we stated a little earlier for the complete integral I in terms of its three components I_1 , I_2 and I_3 , we can now give our solution to the original integral as follows:

$$I = \frac{1}{9} \left(7 \ln|x-2| - 7 \ln|x+1| + \frac{12}{x+1} \right) + C$$

(We should note that the minus sign which precedes the I_3 term in our formula for I in terms of its components has been changed to a plus sign by the minus sign in the solution to the integral I_3 .)

We can simplify this by applying the rules of logarithms to the difference of two logarithms inside the brackets, each of which share a common factor of 7. This gives us the following neater final solution:

$$I = \frac{1}{9} \left(7 \ln \left| \frac{x-2}{x+1} \right| + \frac{12}{x+1} \right) + C$$

Integrating the Two-Term Expansion

To conclude this example with the thoroughness which seems to be its due, we ought to look at what happens when we try to integrate the two-term expansion of our integrand which we derived as a first pass at a partial fractions expansion. Recalling what this two-term version was, we note that it tells us the following about the integration of the original expression in this example:

$$I = \int \frac{x+5}{(x-2)(x+1)^2} dx = \frac{1}{9} \int \left(\frac{7}{x-2} - \frac{7x+19}{(x+1)^2} \right) dx$$

We would expect to integrate the right-hand side above term by term, so in line with our earlier procedure, we will divide it up into two separate integrals as follows:

$$I_1 = \int \frac{7}{x-2} dx \text{ and } I_2 = \int \frac{7x+19}{(x+1)^2} dx$$

This will mean that our final solution will be made up of these two integrals using the following formula:

$$I = \frac{1}{9} (I_1 - I_2)$$

The first of these can be dealt with very quickly: using the rule of integration we have brought into play again and again in this section, we can write down its solution as follows:

$$I_1 = 7 \ln|x-2| + C$$

The second integral, I_2 , is of the form:

$$\int \frac{ax + b}{(cx + d)^p} dx \text{ or alternatively } \int (ax + b)(cx + d)^{-p} dx$$

which is a type of integral which we were able to subdue using substitution, as demonstrated in [Part 1 Section 8](#).

Accordingly, we will deal with the integral I_2 by making the substitution:

$$u = x + 1 \text{ so that } \frac{du}{dx} = 1 \text{ and } dx = du$$

We also note that simple rearrangement lets us say that:

$$x = u - 1$$

This can be used to restate the numerator of our integrand as follows:

$$7x + 19 = 7(u - 1) + 19 = 7u + 12$$

Putting all this into place, we have the following integral in terms of u :

$$\int \frac{7u + 12}{u^2} du = \int \frac{7}{u} du + \int 12u^{-2} du$$

The right-hand side here integrates very easily to give the following solution in terms of u :

$$7 \ln|u| - 12u^{-1} + C \text{ or } 7 \ln|u| - \frac{12}{u} + C$$

We can now reverse the substitution to obtain the final solution for I_2 , which is:

$$I_2 = 7 \ln|x + 1| - \frac{12}{x + 1} + C$$

Applying our formula for the solution to the integral I in terms of its component parts, we can see that we have arrived at the solution:

$$I = \frac{1}{9} \left(7 \ln|x - 2| - 7 \ln|x + 1| + \frac{12}{x + 1} \right) + C$$

which reduces exactly as it did the first time we obtained it to give:

$$I = \frac{1}{9} \left(7 \ln \left| \frac{x - 2}{x + 1} \right| + \frac{12}{x + 1} \right) + C$$

so that we find ourselves with the same solution as the one we obtained using the three-term partial fractions expansion of our integrand.

Example 2

In the next example in this section, we will attempt to evaluate the following integral:

$$I = \int \frac{6x^2 + x + 13}{(x-1)(x^2 + 4)} dx$$

Can we deal with the integrand using the approach which has worked for us in the preceding examples? This would require us to obtain values for A and B where:

$$\frac{6x^2 + x + 13}{(x-1)(x^2 + 4)} = \frac{A}{x-1} + \frac{B}{x^2 + 4}$$

Adding the fractions on the right-hand side using a common denominator and then equating numerators will give us the following equation:

$$A(x^2 + 4) + B(x - 1) = 6x^2 + x + 13$$

Setting $x = 1$ will give us this equation for the value of A :

$$5A = 20 \text{ so that } A = 4$$

We can't choose a real value of x which makes the quadratic factor multiplying A take a zero value, since this would require that $x^2 = -4$! So our preferred technique for finding the values of A and B has been stopped in its tracks.

Perhaps we should try the simultaneous-equation approach in this case. To do this, we must first multiply out the left-hand side of the equation we obtained by equating numerators as follows:

$$A(x^2 + 4) + B(x - 1) = Ax^2 + Bx + 4A - B$$

This gives us the following new version of the equation of the two numerators:

$$Ax^2 + Bx + 4A - B = 6x^2 + x + 13$$

Equating coefficients then gives us the following conclusions:

$$A = 6, B = 1 \text{ and } 4A - B = 13$$

Pausing just long enough to notice that this gives us a different value for A from the value we obtained by the method of selecting a useful value for x a few lines previously, we can also see an inconsistency in these three pieces of information: if $A = 6$ and $B = 1$, then $4A - B$ will come to a value of 23, instead of 13 as our final equating of coefficients predicted.

We see that we have inconsistencies right across the board here. When we obtain inconsistent results using reasoning which is usually expected to work, we are forced to the conclusion that our original assumption must be incorrect. Taking particular note of what

happened when we tried equating coefficients in the numerators, we might proceed to try to express our integrand not in the simple form we had initially hoped for, but instead in the following form:

$$\frac{6x^2 + x + 13}{(x - 1)(x^2 + 4)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 4}$$

Here, we will have a linear expression in x as the numerator of our second fraction, instead of the simple constant which we were hoping for previously. Indeed, we might expect this to be more appropriate since the denominator of the second fraction is itself a quadratic, rather than a simple linear expression.

Let's express our expanded version as a single fraction with a common denominator and equate numerators, as we have grown used to doing. Our expansion becomes:

$$\frac{A(x^2 + 4) + (Bx + C)(x - 1)}{(x - 1)(x^2 + 4)}$$

Since we are still unable to choose a real value for x which will make the factor multiplying A take a zero value, we will immediately proceed to use the method of equating coefficients in the numerators. To this end, multiplying out the numerator here will give us:

$$Ax^2 + 4A + Bx^2 - Bx + Cx - C = (A + B)x^2 + (C - B)x + 4A - C$$

and equating numerators produces the following equation:

$$(A + B)x^2 + (C - B)x + 4A - C = 6x^2 + x + 13$$

Finally, equating coefficients will now yield the following three equations in three variables:

$$\begin{aligned} A + B &= 6 \\ C - B &= 1 \\ 4A - C &= 13 \end{aligned}$$

We can rearrange the second of these to express C in terms of B as follows:

$$C = B + 1$$

Substituting this into the third equation gives:

$$4A - B - 1 = 13 \text{ so that } 4A - B = 14$$

Let's now combine this with the first equation from our set of three to produce two equations in two variables:

$$\begin{aligned} A + B &= 6 \\ 4A - B &= 14 \end{aligned}$$

The first equation of this pair rearranges to give:

$$B = 6 - A$$

And we can substitute this into the second equation to give:

$$4A - 6 + A = 14 \text{ so that } 5A = 20 \text{ and } A = 4$$

We are almost home and dry, since we have already derived simple expressions for both B and C . Working through these, we arrive at the following results:

$$\begin{aligned} B &= 6 - A \text{ so that } B = 2 \\ C &= B + 1 \text{ so that } C = 3 \end{aligned}$$

These conclusions allow us to restate our integrand as follows:

$$\frac{6x^2 + x + 13}{(x-1)(x^2 + 4)} = \frac{4}{x-1} + \frac{2x+3}{x^2 + 4}$$

This time, by trying to find a solution where the numerator of our second term is a linear expression rather than just a constant, we have succeeded in finding values for our placeholders A , B and C . We have, however, had to do this by solving three simultaneous equations, since we were unable to make use of the method of selecting appropriate values for x to obtain separate equations for each of the placeholders. (But notice that this time – because we had the numerator of our second fraction in the correct form – this usual method did actually predict the final value for A .)

So far, so good – we are now left with the job of actually carrying out the integration. Having divided the original integrand into two fractional expressions, we will naturally hope to be able to integrate it term by term. To this end, let's use some notation to keep track of our progress. We have noticed that our integral can be split up into two terms, so that:

$$I = I_1 + I_2, \text{ where } I_1 = \int \frac{4}{x-1} dx \text{ and } I_2 = \int \frac{2x+3}{x^2 + 4} dx$$

The first of these integrals can be written down almost immediately, based on the experience we have gained in our previous examples; we have:

$$I_1 = 4 \ln|x-1| + C$$

The second one, however, looks a little less of a pushover. We might approach it best by again dividing it into two parts, since the integrand can be expressed in the following way:

$$\frac{2x+3}{x^2 + 4} = \frac{2x}{x^2 + 4} + \frac{3}{x^2 + 4}$$

We can therefore express our second integral in the form:

$$I_2 = I_3 + I_4, \text{ where } I_3 = \int \frac{2x}{x^2 + 4} dx \text{ and } I_4 = \int \frac{3}{x^2 + 4} dx$$

The integral I_3 shouldn't present us with too much difficulty; it actually comprises a perfect function-derivative pair, and as such is an ideal candidate for the method of substitution. In order to avoid a long blow-by-blow account, however, I will simply appeal to the rule of integration we established in [Part 1 Section 7](#) which states that:

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

where in our present case:

$$f(x) = x^2 + 4 \text{ and } f'(x) = 2x$$

This lets us write down our solution straight away, so that:

$$I_3 = \ln(x^2 + 4) + C$$

(Notice that we can dispense with our usual modulus brackets in this instance since $x^2 + 4$ will always be positive, because x^2 is always positive.)

The final integral, I_4 is rather less familiar in the current context. It is, however, an example of a standard integral we have encountered before in this book – the integral in question is this one:

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{1}{a} x + C$$

In the present case, $a^2 = 4$, so that $a = 2$, and we can once again write down our solution immediately, as follows:

$$I_4 = \frac{3}{2} \tan^{-1} \frac{1}{2} x + C$$

We have now finished the bulk of our task. It simply remains to tack together the various component parts of our final solution. First, $I_2 = I_3 + I_4$, which means that:

$$I_2 = \ln(x^2 + 4) + \frac{3}{2} \tan^{-1} \frac{1}{2} x + C$$

and finally, $I = I_1 + I_2$, so that:

$$I = 4 \ln|x - 1| + \ln(x^2 + 4) + \frac{3}{2} \tan^{-1} \frac{1}{2} x + C$$

This is, of course, slightly unusual as a solution to a job of integration using partial fractions; however, the unusual element – an inverse tangent function at the end of the solution – is just a natural consequence of the presence of $x^2 + 4$ in the denominator of the second fractional expression in our expansion of the integrand. I might also mention another example in this book, where logarithms and inverse tangents appear in an alternating fashion in the various solutions of the class of related integrals:

$$I_n = \int \frac{x^n}{x^2 + 1} dx$$

which are investigated in [Appendix 5 Section 9](#).

Example 3

We conclude this section by attempting an example unlike anything we have seen so far here (at least, that may be our first impression). We will look at the integral:

$$I = \int \frac{1}{x^3 - 1} dx$$

This differs from all of our previous examples in that the denominator of the integrand is a *cubic* expression, and not presented in the form of a product of linear or quadratic terms. If we are to use the Method of Partial Fractions to get this into a form where we can carry out the integration, then we will have to divide this denominator into at least two factors.

From an inspection of the denominator as it stands, we might think that it could be expressed as a product of the simple linear factor $x - 1$ with some unknown quadratic. To see if this speculation might bear fruit (and, incidentally, to find the unknown quadratic in question at the same time) let's equate the product of $x - 1$ and the general quadratic $ax^2 + bx + c$ with the denominator as we have it, and see if we can deduce any consequences of this equation. We will therefore write:

$$(x - 1)(ax^2 + bx + c) = x^3 - 1$$

We might expect to get some useful information about the unknown coefficients a , b and c by multiplying out the left-hand side and then equating coefficients of like powers of x . Let's start by working out what the left-hand side would give us. It multiplies out to produce:

$$ax^3 + bx^2 + cx - ax^2 - bx - c$$

Gathering together like terms and equating the result with our original denominator produces this equation:

$$ax^3 + (b - a)x^2 + (c - b)x - c = x^3 - 1$$

Equating coefficients gives us two immediate consequences:

$$a = 1 \text{ and } -c = -1 \text{ so that } c = 1$$

So far, so good – we still have two other pieces of information, which constitute requirements that must be satisfied for our proposed expansion to be possible. These are the equations of the two remaining coefficients, which are:

$$b - a = 0 \text{ and } c - b = 0$$

These arise because the denominator on the right-hand side of our equation contains no terms in either x^2 or x , so that the coefficients corresponding to these terms must be zero on the right-hand side.

Substituting into the above two equations our necessary values for a and c , the first of these two equations of coefficients gives us:

$$b - 1 = 0 \text{ so that } b = 1$$

Is this consistent with the second equation of coefficients? It gives us:

$$1 - b = 0 \text{ so that, once again, } b = 1$$

Fortunately, then, we have a set of values for a , b and c which seem completely consistent. (If equating coefficients above had given us inconsistent results, we would, of course, have had to conclude that no factorisation of our desired type was possible.) The useful end result of this reasoning, therefore, is that we can factorise the numerator of our integrand as follows:

$$x^3 - 1 = (x - 1)(x^2 + x + 1)$$

(I pause here to remark that this expansion may have been obvious already to many readers, who will have learnt this result as a standard rule of algebra. It may even have seemed to some no more than a special case of the more general rule:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

As always in the course of this book, however, I have presented a derivation for the benefit of those who do not know either of these results as standard algebraic rules.)

To return to the main plot, however: the factorisation we now have should mean that we can express our integrand in the following form:

$$\frac{1}{x^3 - 1} = \frac{1}{(x - 1)(x^2 + x + 1)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1}$$

This proposed expansion would be in line with the partial fractions expansion we found necessary in the previous example, where the original denominator was a product of one linear and one quadratic factor.

As before, we will now re-express our proposed expansion, complete with its placeholders, as one single fraction with the common denominator $(x - 1)(x^2 + x + 1)$ and then equate the numerator with our original numerator of 1. Our expansion in this restated form would have the following numerator:

$$A(x^2 + x + 1) + (Bx + C)(x - 1)$$

This multiplies out to give the following:

$$Ax^2 + Ax + A + Bx^2 - Bx + Cx - C$$

Gathering together like terms and equating with our original numerator, we arrive at:

$$(A + B)x^2 + (A - B + C)x + A - C = 1$$

Equating coefficients in the way we have become accustomed to do produces the following three equations in A , B and C :

$$\begin{aligned}A + B &= 0 \\A - B + C &= 0 \\A - C &= 1\end{aligned}$$

We can solve these quite easily. The third equation gives us the following expression for C in terms of A alone:

$$-C = 1 - A \text{ so that } C = A - 1$$

Substituting this into the second equation, we have:

$$A - B + A - 1 = 0 \text{ so that } 2A - B = 1$$

By combining this with the first of our three equations, we now have the following two equations in A and B :

$$\begin{aligned}A + B &= 0 \\2A - B &= 1\end{aligned}$$

The first of these can be rearranged to give:

$$B = -A$$

and this can be substituted into the second equation to give:

$$2A + A = 1 \text{ so that } 3A = 1 \text{ and } A = \frac{1}{3}$$

The expression for B obtained above then produces:

$$B = -\frac{1}{3}$$

and finally the expression for C in terms of A which we obtained a little earlier then yields:

$$C = \frac{1}{3} - 1 = -\frac{2}{3}$$

From these conclusions, we now have the following expansion for our integrand:

$$\frac{1}{x^3 - 1} = \frac{1}{3} \left(\frac{1}{x - 1} - \frac{x + 2}{x^2 + x + 1} \right)$$

made a little less inelegant by taking outside the brackets the common factor of $1/3$ shared by all the values we have found for our placeholders.

Can we now proceed to integrate this expansion? Well, we can first remark that our integral I can now be expressed as follows:

$$I = \frac{1}{3} \left(\int \frac{1}{x-1} dx - \int \frac{x+2}{x^2+x+1} dx \right)$$

We have thus divided our integral into two separate integrals, which, as we have done before, we will name I_1 and I_2 , where:

$$I_1 = \int \frac{1}{x-1} dx \text{ and } I_2 = \int \frac{x+2}{x^2+x+1} dx$$

Our final solution will be obtained by combining the solutions to these two integrals using the following formula:

$$I = \frac{1}{3} (I_1 - I_2)$$

which reflects by means of the shorthand notation we have adopted the shape of the two-integral formula for I which we were able to set out thanks to our use of the Method of Partial Fractions to derive a two-term expansion for the integrand we began with.

The first of these two integrals is, by now, easy. We can write down its solution straight away using the by now familiar rule of integration which we have applied again and again in this section. Its solution is:

$$I_1 = \ln|x-1| + C$$

The second integral, I_2 , however, is not so simple. We have encountered an integral of this type already in this book, however, in [Part 1 Section 12](#). There, we found that we could complete the square for the quadratic in the denominator, and then proceed by substitution, and so that is what we will do here.

Completing the square for the denominator (using the formula for a monic quadratic which is given in [Appendix 3 Section 1](#) so as to sidestep a detailed explanation in the current context) gives us the following:

$$x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

Not perhaps the most convenient result, but one we can use nonetheless. We will go on to make the substitution:

$$u = x + \frac{1}{2} \text{ so that } x = u - \frac{1}{2} \text{ and } dx = du$$

This gives us the following integral in terms of u :

$$\int \frac{u + \frac{3}{2}}{u^2 + \frac{3}{4}} dx$$

We might at this point be tempted to multiply the numerator and denominator of the integrand in u by a factor of 4, in order to get rid of the fractional terms. However, we will resist this temptation, in order to follow through the procedure of [Part 1 Section 12 Example 2](#), where we successfully integrated a function of the kind we have to deal with here. As in that example, we will now divide the integral in terms of u into two separate integrals, which we will name I_3 and I_4 , as follows:

$$I_3 = \int \frac{u}{u^2 + \frac{3}{4}} du \text{ and } I_4 = \frac{3}{2} \int \frac{1}{u^2 + \frac{3}{4}} du \text{ so that } I_2 = I_3 + I_4$$

I will go through the completion of this integration process fairly quickly, appealing to the reasoning already given in [Part 1 Section 12 Example 2](#) for any detailed justification which may seem necessary.

Accordingly, we note that the integrand in I_3 is of the classic form to make it suitable for integration by substitution – it has the form:

$$kg'(f(u))f'(u)$$

where:

$$g'(f(u)) = \frac{1}{u^2 + \frac{3}{4}}; f(u) = u^2 + \frac{3}{4}; f'(u) = 2u; k = \frac{1}{2}$$

The solution can therefore be written down almost immediately – either using reasoning exactly parallel to that used in the example from [Part 1](#), or by quoting the formula we established in [Part 1 Section 5](#), to the effect that the solution will have the form:

$$kg(f(x)) + C$$

so that in this case:

$$I_3 = \frac{1}{2} \ln \left(u^2 + \frac{3}{4} \right) + C$$

In the case of I_4 , we will apply the standard integral set out in [Appendix 4 Section 2](#):

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{1}{a} x + C$$

Taking into account the coefficient of $3/2$ outside the integral sign, and noting that in applying the standard integral to our present integral, we have:

$$a^2 = \frac{3}{4} \text{ so that } a = \frac{\sqrt{3}}{2}$$

this gives us the following solution to I_4

$$I_4 = \frac{3}{2} \cdot \frac{2}{\sqrt{3}} \tan^{-1} \frac{2}{\sqrt{3}} u + C = \frac{3}{\sqrt{3}} \tan^{-1} \frac{2}{\sqrt{3}} u + C = \sqrt{3} \tan^{-1} \frac{2}{\sqrt{3}} u + C$$

We will now remind ourselves that $I_2 = I_3 + I_4$, and apply this formula to obtain the solution to I_2 in terms of u :

$$I_2 = \frac{1}{2} \ln \left(u^2 + \frac{3}{4} \right) + \sqrt{3} \tan^{-1} \frac{2}{\sqrt{3}} u + C$$

We must now remind ourselves (since there has been quite a lot to think about since we first established it) that the u -substitution which we made to arrive at this solution was:

$$u = x + \frac{1}{2}$$

So that on reversing this substitution, we now have the following solution for I_2 in terms of x :

$$I_2 = \frac{1}{2} \ln \left(\left(x + \frac{1}{2} \right)^2 + \frac{3}{4} \right) + \sqrt{3} \tan^{-1} \frac{2}{\sqrt{3}} \left(x + \frac{1}{2} \right) + C$$

A further reminder will be necessary here, to the effect that:

$$\left(x + \frac{1}{2} \right)^2 + \frac{3}{4} = x^2 + x + 1$$

(Indeed, this was the equation we obtained when completing the square for the denominator of our original integrand.)

This means that our solution for I_2 can be brought more in line with our original integrand by restating it as follows:

$$I_2 = \frac{1}{2} \ln(x^2 + x + 1) + \sqrt{3} \tan^{-1} \frac{2}{\sqrt{3}} \left(x + \frac{1}{2} \right) + C$$

As the final step in this rather long journey, we will also remind ourselves that:

$$I = \frac{1}{3} (I_1 - I_2)$$

so that our final solution looks like this:

$$I = \frac{1}{3} \left(\ln|x - 1| - \frac{1}{2} \ln(x^2 + x + 1) - \sqrt{3} \tan^{-1} \frac{2}{\sqrt{3}} \left(x + \frac{1}{2} \right) \right) + C$$

Example 4

The last in this series of increasingly sophisticated integration problems which we will solve using partial fractions – the *pièce de résistance*, as it were – will be require us to find a solution to the following integral:

$$I = \int \frac{x-1}{(x-2)(x^2-4x+5)^2} dx$$

The integrand here is another rational function, but this time one which contains a *squared irreducible quadratic* factor in its denominator. It thus contains features from both [Example 1](#) of this section (whose denominator contained a squared, or repeated, linear factor) and [Example 2](#) of this section (whose denominator contained an explicit quadratic as a factor). We might expect, therefore, that we will see how best to decompose this integrand into partial fractions by reminding ourselves of the expansions we obtained in those two examples.

First, [Example 1](#) gave us this expansion:

$$\frac{x+5}{(x-2)(x+1)^2} = \frac{7}{9} \cdot \frac{1}{x-2} - \frac{7}{9} \cdot \frac{1}{x+1} - \frac{4}{3} \cdot \frac{1}{(x+1)^2}$$

and in [Example 2](#) we found the following expansion:

$$\frac{6x^2+x+13}{(x-1)(x^2+4)} = \frac{4}{x-1} + \frac{2x+3}{x^2+4}$$

The salient points arising from these two expansions are these: where the denominator contained a squared factor, it generated two partial fractions – one with the factor squared as denominator, and one with the factor not raised to a power as denominator; and second, where an irreducible quadratic factor was present in the original denominator, it generated a partial fraction with a linear expression of the form we described as $Bx + C$ in its numerator.

The first of the expansions reviewed above would seem to indicate that we should aim to expand our current integrand in the following way:

$$\frac{x-1}{(x-2)(x^2-4x+5)^2} = \frac{A}{x-2} + \frac{P}{x^2-4x+5} + \frac{Q}{(x^2-4x+5)^2}$$

where P and Q represent expressions of a type to be decided. We can, in fact, decide on their nature fairly quickly, since the second expansion quoted above would seem to indicate that P , at least, being the numerator of a fraction with a quadratic in the denominator, should be a linear expression in x . We will also try assuming that the third numerator Q a linear expression as well, and see if we can then find a suitable partial fractions expansion.

All this means, therefore, that we would expect to expand our integrand to a sum of three fractional expressions in the following way:

$$\frac{x-1}{(x-2)(x^2-4x+5)^2} = \frac{A}{x-2} + \frac{Bx+C}{x^2-4x+5} + \frac{Dx+E}{(x^2-4x+5)^2}$$

As we have done in all our recent examples, we will now evaluate the numerator of a single-fraction version of the above, using the common denominator $(x-2)(x^2-4x+5)^2$. We will then equate this with the numerator of our original integrand, and finally obtain values for our placeholders (this time A to E - rather more than we have had to evaluate so far).

The numerator of a single-fraction representation of our desired expansion would therefore be made up as follows:

$$A(x^2-4x+5)^2 + (Bx+C)(x-2)(x^2-4x+5) + (Dx+E)(x-2)$$

In order to put this into a form with all like terms gathered together, which will then let us equate coefficients with our original numerator, we must first multiply out the two compound expressions in x which this expression contains. These are:

$$(x^2-4x+5)^2 = (x^2-4x)^2 + 10(x^2-4x) + 25$$

which multiplies out further to give:

$$x^4 - 8x^3 + 16x^2 + 10x^2 - 40x + 25 = x^4 - 8x^3 + 26x^2 - 40x + 25$$

and:

$$(x-2)(x^2-4x+5) = x^3 - 4x^2 + 5x - 2x^2 + 8x - 10$$

which simplifies to give:

$$x^3 - 6x^2 + 13x - 10$$

Having done this intermediate working, we can then restate our numerator with placeholders in the following way:

$$A(x^4 - 8x^3 + 26x^2 - 40x + 25) + (Bx+C)(x^3 - 6x^2 + 13x - 10) + (Dx+E)(x-2)$$

We will multiply out this full expression in three stages, corresponding to the three terms it contains. These multiply out to give the following three expressions – first:

$$Ax^4 - 8Ax^3 + 26Ax^2 - 40Ax + 25A$$

Second:

$$Bx^4 - 6Bx^3 + 13Bx^2 - 10Bx + Cx^3 - 6Cx^2 + 13Cx - 10C$$

which simplifies further to the following:

$$Bx^4 + (C-6B)x^3 + (13B-6C)x^2 + (13C-10B)x - 10C$$

and finally:

$$Dx^2 - 2Dx + Ex - 2E = Dx^2 + (E - 2D)x - 2E$$

We can now gather together like terms by adding these three expressions together term by term. This gives us the following coefficients for each power of x in our numerator expression:

$$\begin{aligned} &(A + B)x^4 \\ &(C - 6B - 8A)x^3 \\ &(26A + 13B - 6C + D)x^2 \\ &(13C - 10B - 40A + E - 2D)x \\ &25A - 10C - 2E \end{aligned}$$

Equating the coefficients as summarised above with the numerator of our integrand, which was $x - 1$ (and therefore has zero coefficients for all powers of x greater than 1), we have the following five equations in our five variables A to E :

$$\begin{aligned} A + B &= 0 \\ C - 6B - 8A &= 0 \\ 26A + 13B - 6C + D &= 0 \\ 13C - 10B - 40A + E - 2D &= 1 \\ 25A - 10C - 2E &= -1 \end{aligned}$$

We have not so far attempted to solve five equations in five variables, but I propose to illustrate a solution here, using the same method which I have adopted on every previous occasion: I will successively obtain an expression for each of the variables from E down to B in terms of the variables preceding them in the alphabet, and this will finally lead to a simple equation for the value of A . Successive expression for B to E in terms of earlier letters will then allow us to read off the remaining values without difficulty.

We will start, therefore, by taking the fourth equation (which contains a term in E with a coefficient of 1) and rearranging it to give the following expression for E in terms of the other variables:

$$E = 40A + 10B - 13C + 2D + 1$$

The only other equation which contains a term in E is the fifth and final one. Substituting our expression for E into this equation gives us:

$$25A - 10C - 80A - 20B + 26C - 4D - 2 = -1$$

and this simplifies to:

$$16C - 55A - 20B - 4D = 1$$

Restating the other three equations and adding this one gives us these four equations in four variables:

$$A + B = 0$$

$$\begin{aligned}C - 6B - 8A &= 0 \\26A + 13B - 6C + D &= 0 \\16C - 55A - 20B - 4D &= 1\end{aligned}$$

The third of these rearranges to give us a simple expression for D :

$$D = 6C - 26A - 13B$$

and this can be substituted into the fourth equation in the last list to give us:

$$16C - 55A - 20B - 24C + 104A + 52B = 1$$

which simplifies to:

$$32B + 49A - 8C = 1$$

We now have the following three equations in three variables:

$$\begin{aligned}A + B &= 0 \\C - 6B - 8A &= 0 \\32B + 49A - 8C &= 1\end{aligned}$$

The second of these rearranges to give us the following expression for C :

$$C = 8A + 6B$$

Substituting this into the third equation in the most recent list gives:

$$32B + 49A - 64A - 48B = 1$$

which simplifies to give:

$$-15A - 16B = 1$$

We can multiply both sides by -1 to remove the opening minus sign and obtain this:

$$15A + 16B = -1$$

We have now progressed to the extent that we have just two equations in two variables. These are:

$$\begin{aligned}A + B &= 0 \\15A + 16B &= -1\end{aligned}$$

As our final step, we rearrange the first equation in this pair to give B in terms of A as follows:

$$B = -A$$

and then substitute this into the second equation of the pair. This produces:

$$15A - 16A = -1 \text{ so that } -A = -1 \text{ or } A = 1$$

We have now actually found an explicit value for A . We can now make use of all the expressions for specific placeholders which we have so far obtained and values for the other placeholders from B to E should emerge happily. As well as our explicit value for A , I list each of these here:

$$\begin{aligned}A &= 1 \\B &= -A \\C &= 8A + 6B \\D &= 6C - 26A - 13B \\E &= 40A + 10B - 13C + 2D + 1\end{aligned}$$

Evaluating each of these in turn, with the advantage of a progressively larger set of available values at each stage, we obtain the following explicit values for the full set of placeholders:

$$\begin{aligned}A &= 1 \\B &= -1 \\C &= 2 \\D &= -1 \\E &= 3\end{aligned}$$

These results mean that we can now write down the linear expressions which we expected to find as the numerators of the second and third fractional expressions in our expansion. The expressions are as follows:

$$\begin{aligned}Bx + C &= -x + 2 = -(x - 2) \\Dx + E &= -x + 3 = -(x - 3)\end{aligned}$$

I have moved the opening minus sign outside a bracketed expression in both cases, so that we don't have an opening minus sign in the numerators of the final expression, but can instead change the plus sign separating each fractional expression to a minus. This gives us the following partial fractions expansion for our integrand:

$$\frac{x - 1}{(x - 2)(x^2 - 4x + 5)^2} = \frac{1}{x - 2} - \frac{x - 2}{x^2 - 4x + 5} - \frac{x - 3}{(x^2 - 4x + 5)^2}$$

(Some textbooks prefer to change the order of the terms in the two linear numerators, which would give us the following expressions:

$$\begin{aligned}Bx + C &= -x + 2 = 2 - x \\Dx + E &= -x + 3 = 3 - x\end{aligned}$$

and produce the following equivalent partial fractions expansion:

$$\frac{x - 1}{(x - 2)(x^2 - 4x + 5)^2} = \frac{1}{x - 2} + \frac{2 - x}{x^2 - 4x + 5} + \frac{3 - x}{(x^2 - 4x + 5)^2}$$

I have chosen to stick with an ordering which has the x terms first, partly on grounds of taste, and partly for consistency with the kinds of expression we have grown used to seeing while examining the current topic.)

Example 4: Carrying Out the Integration

Now we have split our integrand up into an expanded version comprising three separate terms. We will now try to integrate each of these terms – the “main event” if you will, to which the previous piece of extended algebra was but a prelude. Integrating term by term, we can now define three separate integrals as follows:

$$I_1 = \int \frac{1}{x-2} dx, I_2 = \int \frac{x-2}{x^2-4x+5} dx \text{ and } I_3 = \int \frac{x-3}{(x^2-4x+5)^2} dx$$

When we have evaluated each of these, we will be able to use their solutions to put together a final solution to the main integral I using the formula:

$$I = I_1 - I_2 - I_3$$

To get us off to an encouraging start, we can, of course, write down the solution to I_1 using the by now familiar rule of integration which lets us say that:

$$I_1 = \ln|x-2| + C$$

The second integral, I_2 , is of the form we examined in [Part 1 Section 12](#), where we evaluated integrals like this one using a multi-stage process. The first stage to dealing with an integrand which had an irreducible quadratic expression for its denominator was to complete the square for that quadratic. In our present case, this process (refer to the formula derived for monic quadratics in [Appendix 3 Section 1](#) if necessary) gives the following result:

$$x^2 - 4x + 5 = (x-2)^2 + 1$$

This means that we now have to integrate the following:

$$\int \frac{x-2}{(x-2)^2 + 1} dx$$

Following the approach of [Part 1 Section 12](#), we will make the substitution $u = x - 2$, which gives us the immediate bonus of providing the simple equivalence $dx = du$. Our integral in terms of u is therefore:

$$\int \frac{u}{u^2 + 1} du$$

While we might note in passing that this contains a function-derivative pair with a hidden constant coefficient, and so is a candidate for a further simple substitution, we might also refer directly to the collection of standard integrals set out in [Appendix 4 Section 2](#), where we will learn that:

$$\int \frac{x}{x^2 + 1} dx = \frac{1}{2} \ln(x^2 + 1) + C$$

along with a reference to [Part 2 Section 7](#), where its solution by substitution is sketched out.

In our present circumstances, then, we can quote this standard integral to give the following solution in terms of u :

$$\int \frac{u}{u^2 + 1} du = \frac{1}{2} \ln(u^2 + 1) + C$$

Reversing the substitution, we obtain the following solution in x :

$$\frac{1}{2} \ln((x - 2)^2 + 1) + C$$

We further note that the expression inside the logarithm function is simply the completed-square version of our original quadratic denominator, so that we might prefer to put this solution into a form closer to that of the original integrand by rewriting it as follows:

$$I_2 = \frac{1}{2} \ln(x^2 - 4x + 5) + C$$

We can shortcut the reasoning set out above, however, by noticing that the original integrand in I_2 is of a form we are familiar with from the work we did in [Part 1](#) on integration by substitution: that is, it has the form:

$$kg'(f(x))f'(x)$$

where:

$$g'(f(x)) = \frac{x - 2}{x^2 - 4x + 5}; g'(x) = \frac{1}{x}$$

$$f(x) = x^2 - 4x + 5; f'(x) = 2x - 4, \text{ so that } \frac{1}{2}f'(x) = x - 2 \text{ and thus } k = \frac{1}{2}$$

Since we can conclude from the above that $g(x) = \ln|x|$, the reasoning of [Part 1 Section 5](#) allows us to write down the solution to I_2 directly as:

$$kg(f(x)) + C = \frac{1}{2} \ln(x^2 - 4x + 5) + C$$

All that remains now is to integrate I_3 . Having noted the remarkable convenience of the numerator $x - 2$ which was present in I_2 , we will immediately restate the integrand of I_3 as follows:

$$\frac{x - 3}{(x^2 - 4x + 5)^2} = \frac{x - 2 - 1}{(x^2 - 4x + 5)^2} = \frac{x - 2}{(x^2 - 4x + 5)^2} - \frac{1}{(x^2 - 4x + 5)^2}$$

Example 4: Integrating I_3

We can then go on to integrate I_3 term by term, dividing it into two separate integrals as follows:

$$I_4 = \int \frac{x-2}{(x^2-4x+5)^2} dx \text{ and } I_5 = \int \frac{1}{(x^2-4x+5)^2} dx$$

noting that we will be able to reconstruct a solution to the integral I_3 using the following formula:

$$I_3 = I_4 - I_5$$

Immediately, we see that I_4 can be treated in the same way as I_2 . We will use the shorthand method here, quoting the general rule for straightforward u -substitution and remarking that in this case, once again we have the standard form for the application of the substitution technique, where:

$$g'(f(x)) = \frac{x-2}{(x^2-4x+5)^2}; g'(x) = \frac{1}{x^2}$$

$$f(x) = x^2 - 4x + 5; f'(x) = 2x - 4, \text{ so that } \frac{1}{2}f'(x) = x - 2 \text{ and thus } k = \frac{1}{2}$$

Clearly, the only difference here is the function $g'(x)$. We can rewrite this as:

$$g'(x) = x^{-2} \text{ so that } g(x) = -x^{-1} = -\frac{1}{x}$$

The solution can now be written down quickly as:

$$I_4 = kg(f(x)) + C = -\frac{1}{2(x^2-4x+5)} + C$$

We now move on to consider I_5 . Here, we will begin by completing the square (as we have done before) for the quadratic in the denominator. We recall that:

$$x^2 - 4x + 5 = (x-2)^2 + 1$$

So our integral becomes:

$$\int \frac{1}{((x-2)^2 + 1)^2} dx$$

As we might expect after completing the square, we will substitute $u = x - 2$ which gives us the simple equivalence $dx = du$ and the following integral in terms of u :

$$\int \frac{1}{(u^2 + 1)^2} du$$

Since we now have an expression in the denominator of the form $u^2 + a^2$, where in this case $a = 1$, we will try tangent-substitution as our next step. We will therefore substitute $\tan v = u$, so that:

$$\frac{du}{dv} = \sec^2 v \text{ and thus } dv = \frac{du}{\sec^2 v}$$

Putting in place these substitutions, the denominator becomes:

$$(\tan^2 v + 1)^2 = (\sec^2 v)^2 = \sec^4 v$$

Taking into account the expression for dv , this gives us the following integral in terms of v :

$$\int \frac{1}{\sec^2 v} dv$$

This may initially leave us scratching our heads and doubting whether we have taken the right approach. However, it is important to bear in mind the definition of $\sec x$, which is as follows:

$$\sec x = \frac{1}{\cos x} \text{ so that } \frac{1}{\sec x} = \cos x \text{ and, of course } \frac{1}{\sec^2 x} = \cos^2 x$$

Our integral in terms of v now becomes:

$$\int \cos^2 v dv$$

We can deal with this using the trigonometric identity which states that:

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

so that:

$$\int \cos^2 v dv = \frac{1}{2} \int (1 + \cos 2v) dv = \frac{1}{2} \left(v + \frac{1}{2} \sin 2v \right) + C$$

We are now almost home and dry. We complete this process by reversing the substitutions – first, returning to an expression in terms of u , we have, since $v = \tan^{-1} u$:

$$\frac{1}{2} \left(\tan^{-1} u + \frac{1}{2} \sin 2 \tan^{-1} u \right) + C$$

As we have done before in these pages, we can simplify the rather ugly expression $\sin 2 \tan^{-1} u$ using the following trigonometric identity derived in [Appendix 4 Section 4](#):

$$\sin 2 \tan^{-1} u = \frac{2u}{u^2 + 1}$$

In order to use this identity to make our solution to the integral I_5 in terms of u a little more elegant, we first remark that the term of our solution which we wanted to improve now looks like this:

$$\frac{1}{2} \sin 2 \tan^{-1} u = \frac{u}{u^2 + 1}$$

So that our solution in terms of u now takes the following form:

$$\frac{1}{2} \left(\tan^{-1} u + \frac{u}{u^2 + 1} \right) + C$$

Finally, reversing the u -substitution with which we began - we may recall that this was $u = x - 2$ - we have the following solution to I_5 in terms of x :

$$\frac{1}{2} \left(\tan^{-1}(x - 2) + \frac{x - 2}{(x - 2)^2 + 1} \right) + C$$

Before we declare ourselves satisfied with this solution, we ought to recall just one more time that:

$$x^2 - 4x + 5 = (x - 2)^2 + 1$$

which was the equation we obtained by completing the square for the quadratic in our original denominator. This lets us write down the following final solution for I_5 :

$$I_5 = \frac{1}{2} \left(\tan^{-1}(x - 2) + \frac{x - 2}{x^2 - 4x + 5} \right) + C$$

The intermediate integral I_3 which we have been evaluating here was defined using the following formula:

$$I_3 = I_4 - I_5$$

In order to apply this formula, we will recall the solutions we have obtained for the two intermediate integrals I_4 and I_5 . These are as follows:

$$I_4 = -\frac{1}{2(x^2 - 4x + 5)} + C$$

$$I_5 = \frac{1}{2} \left(\tan^{-1}(x - 2) + \frac{x - 2}{x^2 - 4x + 5} \right) + C$$

We can now put these together to obtain a solution for I_3 using our formula. Showing the coefficients of $1/2$ which seem to be present in both intermediate integrals as they apply to each individual term, our first draft of a solution to I_3 will be as follows:

$$-\frac{1}{2} \cdot \frac{1}{x^2 - 4x + 5} - \frac{1}{2} \tan^{-1}(x - 2) - \frac{1}{2} \cdot \frac{x - 2}{x^2 - 4x + 5} + C$$

The two rational expressions in the above solution can be added together (since they have the same denominator) to produce the following, once we factorise out the common factor of $-1/2$:

$$-\frac{1}{2} \cdot \frac{1+x-2}{x^2-4x+5} = -\frac{1}{2} \cdot \frac{x-1}{x^2-4x+5}$$

This lets us rewrite our solution for I_3 as:

$$I_3 = -\frac{x-1}{2(x^2-4x+5)} - \frac{1}{2}\tan^{-1}(x-2) + C$$

Example 4: Putting the Parts Together

All we now have to do is to put together the results of the various integrations we have just carried out in order to find the overall result for the original integral I . We will first recall the formula we set out when we decomposed the original integral into three parts we have now dealt with.

The overall solution to the original integral I can now be constructed using the following formula:

$$I = I_1 - I_2 - I_3$$

We must now recall our earlier solutions to the intermediate integrals I_1 , I_2 and I_3 . These are:

$$I_1 = \ln|x-2| + C$$

$$I_2 = \frac{1}{2}\ln(x^2-4x+5) + C$$

$$I_3 = -\frac{x-1}{2(x^2-4x+5)} - \frac{1}{2}\tan^{-1}(x-2) + C$$

Our formula for the solution to the original integral I can now be applied to give us this solution:

$$\ln|x-2| - \frac{1}{2}\ln(x^2-4x+5) + \frac{x-1}{2(x^2-4x+5)} + \frac{1}{2}\tan^{-1}(x-2) + C$$

(We note that the minus signs present in the intermediate solution to I_3 have been turned into plus signs by the minus sign preceding I_3 in our formula.)

Reordering the terms of the above expression to bring the pure rational expression to the front (since this was the form of the original integral) we can now write down our final solution as follows:

$$I = \frac{x-1}{2(x^2-4x+5)} + \ln|x-2| - \frac{1}{2}\ln(x^2-4x+5) + \frac{1}{2}\tan^{-1}(x-2) + C$$

3 Summary of Partial Fractions Applied to Integration

It will be worthwhile at this point to summarise what we have learnt by example in the two preceding sections about how we can use the Method of Partial Fractions to help us integrate some rational expressions.

I will divide this summary into three portions, each relating to the simplification of expressions with a particular class of denominator.

Denominators with Linear Factors

The general form of the rational functions of this type which we dealt with in [Section 1](#) above took one of the following two types:

$$\frac{px+q}{(x+a)(x+b)} \text{ or } \frac{px+q}{(x+a)(x+b)(x+c)}$$

Most of our examples only had two linear factors in the denominator, but we did deal with an example containing a three-factor denominator in [Section 1 Example 4](#).

In [Section 1 Example 1](#), the coefficient p in the numerator had a value of zero, so that the numerator comprised a constant term only. [Examples 2 and 3](#), however, did exhibit numerators of the more general linear form.

We found that rational expressions of this form could be expanded into the alternative forms:

$$\frac{A}{x+a} + \frac{B}{x+b} \text{ or } \frac{A}{x+a} + \frac{B}{x+b} + \frac{C}{x+c}$$

depending on the number of linear factors in the denominator, and where A , B and C are numeric constants.

The constants A , B and C were determined using one of two alternative approaches, both requiring us to obtain an expression for the numerator which would appear if we were to combine the expanded version into one single rational expression. This numerator, which would be presented containing instances of the placeholders A , B and C , was then equated with the numerator of the original rational expression.

In the case of the rational expression with two factors in its denominator, the two-term expansion would take the following form if we were to rearrange it as one single expression with a common denominator:

$$\frac{A(x+b) + B(x+a)}{(x+a)(x+b)}$$

Equating the numerator of this expression with the numerator we started out with would give us the following equation:

$$A(x + b) + B(x + a) = px + q$$

The first method (and by far the simpler of the two) we have used to determine values for the placeholders A and B requires us to note that the above equation must be true for all possible real values of x . We can expect, therefore, to be able to select an arbitrary value for x which just happens to make one of the bracketed factors multiplying one of the placeholders take a value of zero. In the above general case, we would select the value $x = -a$, which reduces our equation of numerators to the following:

$$A(b - a) = q - ap$$

by making the factor $x + a$ become zero. This equation can be easily solved to obtain a value for A - note that in a practical situation, we would obtain an equation of the form:

$$rA = s$$

where r and s are numbers. For those interested, the general solution for A in our symbolic case would of course be:

$$A = \frac{q - ap}{b - a}$$

An exactly similar approach can be taken to find the value of B , in our symbolic case by selecting $x = -b$

The second method which we used requires us to multiply out the left-hand side of our equation of numerators so as to above at the following restated equation:

$$Ax + bA + Bx + aB = px + q$$

We can gather like terms together to obtain this version:

$$(A + B)x + bA + aB = px + q$$

By now simply equating coefficients (ie equating the coefficients of x and also equating the constant terms) we will obtain two simultaneous equations in A and B - in our symbolic case, these will look like this:

$$\begin{aligned} A + B &= p \\ bA + aB &= q \end{aligned}$$

In practical cases, where everything in these equations except for A and B is a number, solving these simultaneous equations will provide us with values for A and B .

Both methods will work equally well in the case of the three-term expansion of a rational expression with three linear factors in the denominator – in particular, I tried to show in my remarks following [Section 1 Example 4](#) that the first method (choosing suitable values for x in order to obtain simple equations for each of the placeholders) should always work in cases where we start out with three factors in the denominator of our rational expression.

In dealing with the more complicated types of rational expression which follow, however, we came to find that the simultaneous equation approach was the most useful – since in some cases, no suitable value of x could be chosen to allow the first method I have described method to work.

A *third method* is sometimes taught in older textbooks, which is closely related to the first method I have used in this book, but a little more sophisticated – and perhaps even a more rapid way of getting results. For completeness, and for the benefit of those who might prefer it, I have given an account of this so-called “*cover-up method*” in [Appendix 3 Section 3](#).

The expanded forms of our rational expressions, once obtained using the methods just set out, could then be integrated term by term using the following rules of integration:

$$\int \frac{1}{x+k} dx = \ln|x+k| + C$$

and, in the case of more general linear factors appearing in the original denominator:

$$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln|ax+b| + C$$

(We note that these rules of integration could be stated in a more useful and relevant form as follows:

$$\int \frac{K}{x+k} dx = K \int \frac{1}{x+k} dx = K \ln|x+k| + C$$

and:

$$\int \frac{K}{ax+b} dx = K \int \frac{1}{ax+b} dx = \frac{K}{a} \ln|ax+b| + C$$

to emphasise that a constant numerator other than 1 is no barrier to their use. Indeed, we saw in [Section 1 Example 3](#) that fractional values of the constant in the numerator don’t prevent us using the rule either.)

We also saw in the last example in [Section 1](#) that expressions with a quadratic numerator and three linear factors in the denominator – that is, expressions of the form:

$$\frac{px^2 + qx + r}{(x+a)(x+b)(x+c)}$$

could also be restated in the form:

$$\frac{A}{x+a} + \frac{B}{x+b} + \frac{C}{x+c}$$

and then integrated term by term like those with linear numerators.

Denominators with Repeated Linear Factors

We began [Section 2](#) above with the integration of an expression of the following form:

$$\frac{ps + q}{(x + a)(x + b)^2}$$

The denominator here can be regarded as containing a repeated factor of $x + b$.

We obtained two different expansions for this kind of expression – one with two terms and one with three. These expansions had the following forms:

$$\frac{A}{x + a} + \frac{Bx + C}{(x + b)^2} \text{ and } \frac{A}{x + a} + \frac{B}{x + b} + \frac{C}{(x + b)^2}$$

In each of these two cases, the final term could not be integrated using the simple rule of integration which had worked for us up to this point, so that different techniques had to be called into play.

The integral:

$$\int \frac{Bx + C}{(x + b)^2} dx$$

was dealt with using the substitution $u = x + b$, and the integration of the final term in the three-term expansion:

$$\int \frac{C}{(x + b)^2} dx$$

was carried out using another of the rules of integration we derived from the method of substitution in [Part 1](#), namely:

$$\int (f(x))^n f'(x) dx = \frac{1}{n + 1} (f(x))^{n+1} + C$$

(In the present case, we note that $n = -2$, a negative power.)

Denominators with Quadratic Factors

[Examples 2 and 3 of Section 2](#) concerned integrands with an irreducible quadratic factor in their denominators. These were of the general form:

$$\frac{px^2 + qx + r}{(x + a)(x^2 + bx + c)}$$

We were able to obtain expansions for these expressions on the following model:

$$\frac{A}{x+a} + \frac{Bx+C}{x^2+bx+c}$$

Two different techniques were used to integrate the second term of this expansion (although the first of these could only be applied because in [Section 2 Example 2](#), the coefficient b was zero). In general, we would have to solve this integral by first completing the square for the quadratic in the denominator and then using substitution as we did in [Part 1 Section 12](#). The resulting integral in terms of u could then be solved by splitting it into two separate terms and using a further substitution or the rule of integration which states that:

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{1}{a} x + C$$

to complete the process.

Denominators with Repeated Quadratic Factors

[Section 2 Example 4](#) took things one last step further by requiring us to integrate an expression with a squared (or “repeated”) irreducible quadratic factor in the denominator. The integrand we dealt with was of the following form:

$$\frac{px+q}{(x+a)(x^2+bx+c)^2}$$

The expansion we managed to produce for the expression in this example was of the following form:

$$\frac{A}{x+2} + \frac{Bx+C}{x^2+bx+c} + \frac{Dx+E}{(x^2+bx+c)^2}$$

Since this expansion contained no less than five placeholders spread across the various numerators, we were confronted with five simultaneous equations in five variables to solve. Once this was done, we were then left with the task of integrating the resulting expansion term by term.

The first term was a prime candidate for the usual rule of integration which we have used repeatedly in the simpler partial fractions examples. The second term was dealt with by completing the square for the denominator and then using substitution. We ought to note that, despite being of a similar form to the second term we obtained in [Section 2 Example 3](#), the different numerator and denominator gave rise to a completely different solution to the integral.

The third term, containing as it did a squared quadratic as its denominator, required a little more thought. We solved the problem by dividing the integral into two terms, and then applying a combination of completing the square, multiple substitutions (including a trigonometric substitution) and the simplification of an expression which combined a trigonometric function and an inverse trigonometric function. This was by far the most challenging integral so far in this book, calling for a combination of techniques and some additional manipulation of trigonometric functions.

4 Power Substitution and Partial Fractions

As a postscript to our fairly detailed look at integrals which required the use of partial fractions expansions in their solution, this section will concern itself with some integrals where power substitution is the first step in their solution, but where we end up having to obtain a partial fractions expansion in order to complete the job. This topic is fully in the spirit of this third part of the book, since we will plainly require more than one technique to solve the problems which are presented here.

In general shape, the integrands in this section will be rational functions which contain roots – it will be these roots which need the application of power substitution. However, the integrands' status as rational functions will require us to call on the Method of Partial Fractions in order to prepare these expressions fully for integration.

Example 1

Our first example will be the following integral – innocent enough in appearance, but, as we shall see, calling for some careful work when we come to evaluate it;

$$I = \int \frac{\sqrt{x}}{x-1} dx$$

The square root in the denominator might be expected to suggest the use of power substitution (as long as we remark beforehand that nothing like a function-derivative pair is in evidence here to point us towards a straightforward u -substitution). We will therefore take the usual step of making an algebraic substitution calculated to remove the square root from our consideration. This substitution will be:

$$x = u^2 \text{ so that } \sqrt{x} = \sqrt{u^2} = u$$

Differentiating x with respect to u will give us the following:

$$\frac{dx}{du} = 2u \text{ so that } dx = 2u du$$

We can now put these results back into the original integral to obtain the following revised integral in terms of u :

$$\int \frac{u}{u^2-1} \cdot 2u du = \int \frac{2u^2}{u^2-1} du$$

Now the integral which remains on the right-hand side above, if we temporarily remove the constant coefficient of 2, is of the form:

$$\frac{x^n}{x^n + a}$$

which we explored in [Appendix 3 Section 2](#) (having encountered this kind of expression when discussing substitution in [Part 1](#)). In the present case, of course, we note that $a = -1$.

So let's see what we can do with the present expression, again removing the coefficient of 2 for the moment (we will be sure to bring it back later!). We can apply the technique from [Appendix 3](#) to “reduce” this rational expression by adding and subtracting 1 in the numerator, as follows:

$$\frac{u^2}{u^2 - 1} = \frac{u^2 - 1 + 1}{u^2 - 1} = \frac{u^2 - 1}{u^2 - 1} + \frac{1}{u^2 - 1} = 1 + \frac{1}{u^2 - 1}$$

So far, so good. Let's be sure to put back our coefficient of 2, which now multiplies the whole of the “reduced” version of our expression, leaving us with the following integrand:

$$2 + \frac{2}{u^2 - 1}$$

Having taken this step, we find that the rational term in our new expression is still not, perhaps, entirely to our taste. However, as we have done before in our look at partial fractions expansions, we can usefully recall at this point the algebraic identity:

$$a^2 - b^2 = (a - b)(a + b)$$

which will allow us to rewrite the denominator of the rational term as the product of two linear factors, as follows:

$$\frac{2}{u^2 - 1} = \frac{2}{(u - 1)(u + 1)}$$

The right-hand side here is ripe for the use of the Method of Partial Fractions, so let's immediately get down to obtaining an expansion of the following form – the most basic form for a partial fractions expansion:

$$\frac{2}{(u - 1)(u + 1)} = \frac{A}{u - 1} + \frac{B}{u + 1}$$

As usual, we want to re-express the right-hand side as a single fraction with the common denominator $(u - 1)(u + 1)$, and then equate its numerator with the numerator of the expression we started with. I hope that by this time I might be excused for cutting straight to the chase and just writing down the equation of numerators which will result; this equation is:

$$A(u + 1) + B(u - 1) = 2$$

Luckily, the position now could hardly be simpler: we will choose values of u which will make each of the two bracketed expressions take the value zero in turn. From these two choices we will obtain simple equations for the values of A and B .

First, then, let's set $u = 1$; this yields:

$$2A = 2 \text{ so that } A = 1$$

Second, we must choose $u = -1$; this gives our second equation:

$$-2B = 2 \text{ so that } B = -1$$

We can now write down our partial fractions expansion in the following way:

$$\frac{2}{(u-1)(u+1)} = \frac{1}{u-1} - \frac{1}{u+1}$$

Returning to our integral in terms of u after this series of necessary manipulations, we can now rewrite the integral in the following way:

$$\int \frac{2u^2}{u^2-1} du = \int \left(2 + \frac{2}{(u-1)(u+1)} \right) du$$

The partial fractions expansion we have obtained now lets us expand this further, into the following form:

$$\int 2 du + \int \frac{1}{u-1} du - \int \frac{1}{u+1} du$$

These three integrals can be dealt with easily – the first using a standard rule of integration, and the second and third by applying the rule of integration:

$$\int \frac{1}{x+k} dx = \ln|x+k| + C$$

which we have used repeatedly in the present context. We can therefore write down our solution in terms of u as follows:

$$2u + \ln|u-1| - \ln|u+1| + C$$

We can tidy this up further by applying the by now familiar rule of logarithms:

$$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$$

from which we obtain this new version of our solution:

$$2u + \ln\left|\frac{u-1}{u+1}\right| + C$$

All that remains is to reverse the substitution. We recall that this was:

$$x = u^2 \text{ so that } u = \sqrt{x}$$

Hence, our final solution in terms of x is found to be the following:

$$I = 2\sqrt{x} + \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| + C$$

Example 2

We will continue with the following more challenging integral, which immediately has a rather threatening look to it:

$$I = \int \frac{\sqrt{4-x}}{x^2} dx$$

Since we have the square root of a linear expression in the denominator, we will immediately try using power substitution to remove the root from consideration. Our substitution, in line with the practice we established in [Part 1 Section 9](#), ought to be as follows:

$$4 - x = u^2 \text{ so that } \sqrt{4-x} = \sqrt{u^2} = u$$

Rewriting our substitution formula to make x the subject gives us:

$$x = 4 - u^2$$

Differentiating this with respect to u produces the following information:

$$\frac{dx}{du} = -2u \text{ so that } dx = -2u du$$

Let's put all these facts back into our original integrand in order to obtain our integral in terms of u . This will be:

$$\int \frac{u}{(4-u^2)^2} \cdot (-2u) du = \int \frac{-2u^2}{(4-u^2)^2} du$$

(Notice that, as we have already seen when using power substitution, the powers of u are already ramping up.)

We will find it convenient and a bit more familiar to note that in the denominator of the integrand:

$$4 - u^2 = -(u^2 - 4)$$

and since the square of a quantity a is always equal to the square of $-a$, we can go further and say that:

$$(4 - u^2)^2 = (u^2 - 4)^2$$

So that the integral can be rewritten as:

$$\int \frac{-2u^2}{(u^2 - 4)^2} du$$

At this stage, we should take a long look at the expression in brackets in the denominator of the integrand, and notice that it is a difference of two squares. We can use this fact to rewrite the integrand as follows:

$$\frac{-2u^2}{(u^2 - 4)^2} = \frac{-2u^2}{((u - 2)(u + 2))^2} = \frac{-2u^2}{(u - 2)^2(u + 2)^2}$$

Now this is a rational expression of the kind we have met with already when working with partial fractions, - in this case with two repeated linear factors in the denominator. Using the appropriate case from the classification of rational functions we set out in [Section 3](#) above, this particular expression should be equivalent to an expansion of the following form:

$$\frac{A}{u - 2} + \frac{B}{(u - 2)^2} + \frac{C}{u + 2} + \frac{D}{(u + 2)^2}$$

Now let's proceed to form a single rational expression from the expansion, using the common denominator of $(u - 2)^2(u + 2)^2$. This will be as follows:

$$\frac{A(u - 2)(u + 2)^2 + B(u + 2)^2 + C(u + 2)(u - 2)^2 + D(u - 2)^2}{(u - 2)^2(u + 2)^2}$$

Equating the numerator of this expression with the numerator of our original integrand in u gives us this equation:

$$A(u - 2)(u + 2)^2 + B(u + 2)^2 + C(u + 2)(u - 2)^2 + D(u - 2)^2 = -2u^2$$

Before going to the trouble of all the necessary multiplying out which we would need before we could equate coefficients, it might be quicker to see how far we can get by choosing useful values of u and then solving the resulting rather simpler equations

It's difficult to see what value of u we could choose to leave us with a simple equation in A , since the other three placeholders do not share a common factor. However, we do find that A , C and D all share a common factor of $u - 2$, so that setting $u = 2$ will make this common factor take a zero value, leaving us with the following equation in B alone:

$$16B = -8 \text{ so that } B = -\frac{1}{2}$$

A similar choice, $u = -2$, will make the factor $u + 2$ take the value zero, leaving us with an equation in D alone. This equation is:

$$16D = -8 \text{ so that } D = -\frac{1}{2}$$

These two results mean that we can now rewrite our equation of numerators as follows:

$$A(u - 2)(u + 2)^2 - \frac{1}{2}(u + 2)^2 + C(u + 2)(u - 2)^2 - \frac{1}{2}(u - 2)^2 = -2u^2$$

We have already remarked that no simple choice of value for u is available to leave us with an easy equation in A alone. The same problem faces us with respect to C . We might expect now to be faced with the necessity of multiplying out all the bracketed expressions in u on the left-hand side of the equation and then equating coefficients of powers of u to determine A and C . However, it would be natural to have an instinctive reluctance to pile straight into a complicated piece of algebra unless it is absolutely necessary.

In this case, I am going to propose an alternative way of determining values for the two remaining placeholders which we have not used before when constructing partial fractions expansions. We have noted in several of the examples presented here that we would expect the equation of numerators we have obtained to hold good for all possible values of the main variable (u in the present example). In order to find the values of B and D , we chose values which would leave us with simple equations in those placeholders alone: we did this by ensuring that our chosen values made the factors multiplying the other placeholders take the value zero.

Let's see how useful it might be to choose other, arbitrary, values for u and examine the resulting versions of our equation of numerators. To start with, let's choose the value $u = 0$; this is a useful starting point, since it makes the arithmetic we have to perform as a result of our choice particularly easy. This value transforms our equation of numerators into:

$$A.(-2).4 - \frac{1}{2}.4 + C.2.4 - \frac{1}{2}.4 = (-2).0$$

Completing the necessary arithmetic, we arrive at the following equation:

$$-8A - 2 + 8C - 2 = 0 \text{ or } 8C - 8A = 4$$

If we take the final step of dividing both sides by -4 , we will be left with this equation:

$$2C - 2A = 1$$

which is clearly an equation in two variables., so that we will need one more equation in A and C in order to determine these values.

Let's now choose the "next simplest" value for u - the value $u = 1$. Our equation of numerators now becomes:

$$A.(-1).9 - \frac{1}{2}.9 + C.3.1 - \frac{1}{2}.1 = (-2).1$$

The various calculations in this equation evaluate to give us the following:

$$-9A - \frac{9}{2} + 3C - \frac{1}{2} = -2 \text{ or } -9A + 3C = 3$$

Dividing both sides by 3 and making a small rearrangement to avoid an opening minus sign, this will give us the following equation in A and C :

$$C - 3A = 1$$

We will now draw together the two simultaneous equations in A and C which we have now obtained by substituting successive values of u into our equation of numerators. These two equations are:

$$\begin{aligned}2C - 2A &= 1 \\ C - 3A &= 1\end{aligned}$$

This pair of simultaneous equations is not difficult to solve, in comparison with some of the herculean efforts we have made previously with more complicated systems of equations. First of all, the second equation of the two can easily be rearranged to give an expression for C in terms of A . This rearrangement is as follows:

$$C = 3A + 1$$

We can then substitute this expression for C into the first equation, to arrive at:

$$6A + 2 - 2A = 1 \text{ or } 4A = -1$$

The solution to this equation is, of course:

$$A = -\frac{1}{4}$$

And by finally substituting this into the expression for C in terms of A which we obtained a little earlier, we have the following as a value for C :

$$C = -\frac{3}{4} + 1 \text{ or, more simply } C = \frac{1}{4}$$

We now have values for all four of the placeholders in our partial fractions expansion. These are, in summary

$$A = -\frac{1}{4}, B = -\frac{1}{2}, C = \frac{1}{4} \text{ and } D = -\frac{1}{2}$$

Putting these values for our placeholders into the framework we established for our partial fractions expansion thus gives us the following expansion for our original integrand:

$$\frac{-2u^2}{(u-2)^2(u+2)^2} = -\frac{1}{4} \cdot \frac{1}{u-2} - \frac{1}{2} \cdot \frac{1}{(u-2)^2} + \frac{1}{4} \cdot \frac{1}{u+2} - \frac{1}{2} \cdot \frac{1}{(u+2)^2}$$

(I have positioned the placeholders as coefficients multiplying each rational term in the expansion both to avoid the ugliness of fractions in numerators, and to prepare the way for what is going to happen next when we set out our approach to the integration of our expanded integrand.)

In order to see how we can now best proceed with the actual task of integration, I will now show how we can divide up this task. I will rewrite the integrand in terms of u in the form in which we first encountered it, and also place the third term in our expansion at the beginning so as to remove the opening minus sign. The integration task now looks like this:

$$\int \frac{-2u^2}{(u^2 - 4)^2} du = \frac{1}{4} \int \frac{1}{u+2} du - \frac{1}{4} \int \frac{1}{u-2} du - \frac{1}{2} \int \frac{1}{(u-2)^2} du - \frac{1}{2} \int \frac{1}{(u+2)^2} du$$

Under this rearrangement, the integrals in first two terms can be evaluated easily, using the same rule of integration which we have applied again and again when working with partial fractions expansions. Their integrals are:

$$\ln|u+2| + C \text{ and } \ln|u-2| + C$$

respectively.

The third and fourth integrals will not respond to this rule. They can, however, be integrated using the rule which states that:

$$\int (f(x))^n f'(x) dx = \frac{1}{n+1} (f(x))^{n+1} + C$$

In the case of the third term, we can apply this rule in terms of u with the following substitutions:

$$f(u) = u - 2, f'(u) = 1 \text{ and } n = -2$$

According to the rule we have chosen, the third integral in the expansion evaluates to:

$$-\frac{1}{1}(u-2)^{-1} + C \text{ or, more neatly, } -\frac{1}{u-2} + C$$

The same rule can be applied to the fourth integral in our expansion, with one alteration, to the effect that this time $f(u) = u + 2$. The fourth integral therefore evaluates to:

$$-\frac{1}{1}(u+2)^{-1} + C \text{ or simply } -\frac{1}{u+2} + C$$

We can now gather all these results together, making sure to apply all the fractional constants which precede each integral in our expansion, and obtain the following solution in terms of u :

$$\frac{1}{4} \ln|u+2| - \frac{1}{4} \ln|u-2| - \frac{1}{2(u-2)} - \frac{1}{2(u+2)} + C$$

The first two terms here have a common coefficient of $1/4$, so we can rewrite them as one single term, using the by now familiar rule of logarithms:

$$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$$

This turns our solution in terms of u into the following neater version:

$$\frac{1}{4} \ln \left| \frac{u+2}{u-2} \right| - \frac{1}{2(u-2)} - \frac{1}{2(u+2)} + C$$

All that now remains (if we can recall that all this partial fractions work arose from the use of a substitution) is to reverse our substitution.

We must first remind ourselves that our substitution was as follows:

$$4 - x = u^2 \text{ so that } u = \sqrt{4 - x}$$

We can therefore now write down the final solution to our original integral in terms of x as follows:

$$I = \frac{1}{4} \ln \left| \frac{\sqrt{4-x} + 2}{\sqrt{4-x} - 2} \right| - \frac{1}{2(\sqrt{4-x} - 2)} - \frac{1}{2(\sqrt{4-x} + 2)} + C$$

Example 3

In this final example, we will, as ever, raise the stakes somewhat and consider an integral of a fairly challenging aspect, namely:

$$I = \int \frac{x^{1/4} + 5}{x - 16} dx$$

The integrand contains a fourth root, so we will start off with a power substitution – expecting, in line with our experience in [Part 1](#) using this technique, that we will quickly accumulate some high powers of u when we carry out the substitution.

As we might expect if we are to remove the awkward fourth root in the numerator of our rational integrand, our power substitution will be:

$$x = u^4 \text{ so that } x^{1/4} = (u^4)^{1/4} = u$$

Differentiating the formula for x with respect to u , as we always do, we obtain:

$$\frac{dx}{du} = 4u^3 \text{ so that } dx = 4u^3 du$$

Let's put these facts back into our original integrand in order to obtain an integral in terms of u . We arrive at the following:

$$\int \frac{u + 5}{u^4 - 16} \cdot 4u^3 du = \int \frac{4u^4 + 20u^3}{u^4 - 16} du$$

Since the prospect of integrating this doesn't exactly fill us with confidence, let's start by dividing the integrand up into two separate fractions, as follows:

$$\frac{4u^4 + 20u^3}{u^4 - 16} = \frac{4u^4}{u^4 - 16} + \frac{20u^3}{u^4 - 16} = 4 \cdot \frac{u^4}{u^4 - 16} + 20 \cdot \frac{u^3}{u^4 - 16}$$

Having dismantled the rational expression in u in this way, we immediately notice that the first rational expression on the final right-hand side above (with its coefficient of 4 taken to one side for a moment) is of the form:

$$\frac{x^n}{x^n + a}$$

where $a = -16$. We recalled in [Example 1](#) of this section that such expressions can be “reduced” in a useful way using the technique discussed in [Appendix 3 Section 2](#). In our particular case, the manipulation called for is as follows:

$$\frac{u^4}{u^4 - 16} = \frac{u^4 - 16 + 16}{u^4 - 16} = \frac{u^4 - 16}{u^4 - 16} + \frac{16}{u^4 - 16} = 1 + \frac{16}{u^4 - 16}$$

At this point, we must remember to reapply the coefficient of 4, which we put on one side temporarily to make the above piece of algebra a little easier.. Taking this step, we can now write out our integrand in terms of u in the following form:

$$\frac{4u^4 + 20u^3}{u^4 - 16} = 4 \left(1 + \frac{16}{u^4 - 16} \right) + \frac{20u^3}{u^4 - 16} = 4 + \frac{64}{u^4 - 16} + \frac{20u^3}{u^4 - 16}$$

This is a three-term expansion, which we can now proceed to deal with on a term-by-term basis. The first term can be polished off very quickly, as follows:

$$\int 4du = 4u + C$$

So much for the first term; let’s now introduce some notation to make the rest of the job a little easier. We will identify the integrals of the two remaining terms in our expanded version of the integrand in u as follows:

$$I_1 = \int \frac{64}{u^4 - 16} du \text{ and } I_2 = \int \frac{20u^3}{u^4 - 16} du$$

With this notation, and once we have actually evaluated these two integrals, we will be able to write down our complete solution in terms of u (which I will name I_u , in order to differentiate it from the solution in terms of x , which we have already identified as I) using the formula:

$$I_u = 4u + I_1 + I_2 + C$$

which is, of course, based on the three-term expansion we set out a little earlier.

Example 3: Integrating I_1

So let’s make our next step the evaluation of the integral; I_1 . You may have already noticed that the denominator is a difference of two squares, so that we can rewrite it as:

$$u^4 - 16 = (u^2 - 4)(u^2 + 4)$$

As an added bonus, the first factor on the right-hand side above is also a difference of two squares, so it can in turn be rewritten as:

$$u^2 - 4 = (u - 2)(u + 2)$$

As a consequence, our quartic denominator can be fully expanded in the following way:

$$u^4 - 16 = (u - 2)(u + 2)(u^2 + 4)$$

All this allows us to rewrite the integrand of I_1 as follows:

$$\frac{64}{u^4 - 16} = \frac{64}{(u - 2)(u + 2)(u^2 + 4)}$$

The right-hand expression is the kind of rational expression we have been expanding using partial fractions – it has two ordinary linear factors and one irreducible quadratic factor. Reference to [Section 3](#) above will confirm that we should therefore be able to expand it as follows:

$$\frac{64}{(u - 2)(u + 2)(u^2 + 4)} = \frac{A}{u - 2} + \frac{B}{u + 2} + \frac{Cu + D}{u^2 + 4}$$

To determine the placeholders, we will as usual restate the right-hand side as one single fractional expression with the common denominator $(u - 2)(u + 2)(u^2 + 4)$ and then equate the numerator with the left-hand side numerator of 64. The equation of numerators will be as follows:

$$A(u + 2)(u^2 + 4) + B(u - 2)(u^2 + 4) + (Cu + D)(u - 2)(u + 2) = 64$$

Right at the outset, we will have to note that we can't find a real value for u which will make the quadratic factor $u^2 + 4$ take a value of zero. This means that we will have to do what we can initially with the two linear factors. To begin with, then, we note to our relief that by choosing the value $u = 2$, we can zero the factor $u - 2$, which will leave us with an equation for A alone. This equation is:

$$32A = 64 \text{ so that } A = 2$$

We will also notice that the second linear factor, $u + 2$, can be set to zero by choosing the value $u = -2$. This leaves us with an equation in B alone, which is:

$$-32B = 64 \text{ so that } B = -2$$

If we now insert these values into our equation of numerators, we will have the following revised equation:

$$2(u + 2)(u^2 + 4) - 2(u - 2)(u^2 + 4) + (Cu + D)(u - 2)(u + 2) = 64$$

In the previous example, we managed to get a little further when we had run out of options to set bracketed factors to zero by choosing other “simple” values of u which might help us. In this case, we notice that, once we have determined A and B , setting $u = 0$ will conveniently take the placeholder C out of consideration, allowing us to obtain an equation in D alone. This equation will be:

$$16 + 16 - 4D = 64 \text{ or } -4D = 32 \text{ so that } 4D = -32 \text{ and } D = -8$$

Let's insert this value for D into our equation of numerators, so that we arrive at the following new equation:

$$2(u + 2)(u^2 + 4) - 2(u - 2)(u^2 + 4) + (Cu - 8)(u - 2)(u + 2) = 64$$

We see that we are now left with only C as the last placeholder as yet undetermined. If we now choose $u = 1$, we should be able to complete the job, obtaining the following equation in C alone:

$$30 + 10 - 3(C - 8) = 64 \text{ or } 40 - 3C + 24 = 64$$

This simplifies to give:

$$64 - 3C = 64 \text{ or } 3C = 0 \text{ so that } C = 0$$

This is a rather surprising result, since we have found in previous examples that there was always a linear numerator to go with an irreducible quadratic denominator. Can it be correct? It would be wise to verify the values we have obtained for our placeholders by starting with the partial fractions expansion they specify and calculating the single rational expression which results. We might usefully summarise the values we have obtained here, as follows:

$$\begin{aligned} A &= 2 \\ B &= -2 \\ C &= 0 \\ D &= -8 \end{aligned}$$

On this basis, our partial fractions expansion would be the following:

$$\frac{2}{u - 2} - \frac{2}{u + 2} - \frac{8}{u^2 + 4}$$

If we were to restate this expansion as a single rational expression with the common denominator $(u - 2)(u + 2)(u^2 + 4)$, our numerator would be as follows:

$$2(u + 2)(u^2 + 4) - 2(u - 2)(u^2 + 4) - 8(u - 2)(u + 2)$$

(This is, of course, the numerator we obtained earlier with the four placeholders present; all we have done in effect is so substitute our four values for the various placeholders.)

To see if this numerator reduces to the numerator of our original rational expression in terms of u , we must first multiply out the products of bracketed expressions. This will lead to the following – once the various powers of u have been arranged in descending order:

$$2(u^3 + 2u^2 + 4u + 8) - 2(u^3 - 2u^2 + 4u - 8) - 8(u^2 - 4)$$

Finally, we multiply the bracketed stretches of the above expression by their respective coefficients and end up with this result:

$$2u^3 + 4u^2 + 8u + 16 - 2u^3 + 4u^2 - 8u + 16 - 8u^2 + 32$$

Most conveniently and agreeably, all the terms in u or some power of u cancel out, so that we are left with:

$$16 + 16 + 32 = 64$$

which is, of course, the numerator of the rational expression in u with which we started out. So we have confirmed that our earlier manipulations using various useful values of u have indeed led us to the correct partial fractions expansion. As a final note to all this, it might be worth remarking that it should perhaps not be so surprising that the placeholder C in our original template turned out to be zero, since if it were not, we would have to have some terms in u on the right-hand side of the equation of numerators – in our current case, this right-hand side contained only a constant.

Back to the task in hand; let's just set down the conclusion of the foregoing discussion:

$$\frac{64}{u^4 - 16} = \frac{64}{(u - 2)(u + 2)(u^2 + 4)} = \frac{2}{u - 2} - \frac{2}{u + 2} - \frac{8}{u^2 + 4}$$

We can therefore now expect to integrate I_1 by integrating the final right-hand side above term by term. The first two terms, by this stage in our discussion of partial fractions, are easily dealt with using our familiar integration rule. Their solutions will be:

$$2 \ln|u - 2| + C \text{ and } 2 \ln|u + 2| + C$$

respectively.

The final term looks a little unfamiliar in the context of partial fractions expansions, but it presents little more resistance than the first two terms did. All that is necessary is to recall the standard integral:

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{1}{a} x + C$$

In the third term of our expansion, we have a constant coefficient of 8 present (which can be moved outside the integral sign until we are finished), and we also have $a = 2$. Our third term therefore integrates quite easily to give us:

$$8 \cdot \frac{1}{2} \tan^{-1} \frac{1}{2} x + C = 4 \tan^{-1} \frac{1}{2} x + C$$

This straightforward reasoning allows us to write down the solution to the intermediate integral I_1 as follows:

$$I_1 = 2 \ln|u - 2| - 2 \ln|u + 2| - 4 \tan^{-1} \frac{1}{2}x + C$$

We might at this stage be tempted to call on our old friend the logarithmic identity:

$$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$$

to rewrite this solution in a briefer form. However, I won't do so here, for more compelling reasons which will become apparent in a short while.

Example 3: Integrating I_2

The second of our two intermediate integrals was as follows:

$$I_2 = \int \frac{20u^3}{u^4 - 16} du$$

Fortunately, this is nowhere near as difficult as it looks. Indeed, it is nothing more than a simple function-derivative pair with a constant coefficient. In the notation of [Part 1](#), it has the form

$$kg'(f(x))f'(x)$$

where:

$$k = 5, g'(f(x)) = \frac{1}{u^4 - 16}, g'(x) = \frac{1}{x} \text{ so that } g(x) = \ln|x|$$

$$f(x) = u^4 - 16, f'(x) = 4u^3$$

We established in [Part 1 Section 5](#) that the solution to integrals of this sort can be quickly written down as:

$$kg(f(x)) + C$$

In our present case, then, using the values listed above, we can write down our solution to I_2 as follows:

$$I_2 = 5 \ln|u^4 - 16| + C$$

Example 3: Assembling the Final Solution

Using the formula for the complete integral in terms of u which we stated before digressing into the technicalities surrounding the two intermediate integrals, we can now set out a (preliminary) solution to the full integral in terms of u (which we named I_u) as follows – and I will gather together all the logarithmic terms for convenience in the simplification which will follow:

$$4u + 2 \ln|u - 2| - 2 \ln|u + 2| + 5 \ln|u^4 - 16| - 4 \tan^{-1} \frac{1}{2}u + C$$

Now this might be considered a perfectly respectable solution to take into the final stage of reversing the substitution. However, there is one further step of simplification we can take, and it involves the use of a rule of logarithms – just not the one we have become so used to using in partial fractions examples and which beckoned temptingly to us at the close of our discussion of the integral I_1 .

First of all, we must recall the full algebraic expansion of the expression $u^4 - 16$ which allowed us to choose the appropriate partial fractions expansion for our rational expression in u when we integrated I_1 ; this expansion was:

$$u^4 - 16 = (u - 2)(u + 2)(u^2 + 4)$$

An immediate consequence of this, highly relevant to the position in which we now stand, is this:

$$\ln|u^4 - 16| = \ln|(u - 2)(u + 2)(u^2 + 4)|$$

We just have to couple this observation with the logarithmic identity:

$$\ln(ab) = \ln a + \ln b$$

to be able to state that:

$$\ln|u^4 - 16| = \ln|u - 2| + \ln|u + 2| + \ln(u^2 + 4)$$

(Note here that since u^2 will never be negative for any real value of u , it certainly follows that $u^2 + 4$ will never be negative either, so I have dispensed with the modulus brackets in the final term above.)

Now, taking the logarithmic terms in our preliminary solution for I_u aside on their own for a moment, we can write these out using the above discussion like this:

$$2 \ln|u - 2| - 2 \ln|u + 2| + 5 \ln|u - 2| + 5 \ln|u + 2| + 5 \ln(u^2 + 4)$$

We can, of course, now gather together some like terms to reach the following version of the above sub-expression:

$$7 \ln|u - 2| + 3 \ln|u + 2| + 5 \ln(u^2 + 4)$$

(It was so that I could make this final rearrangement that I demurred when presented with the opportunity to amalgamate the two logarithmic terms in the solution to I_1 into one single term. We remark that now, of course, no such amalgamation is possible due to the differing coefficients of the various logarithmic terms.)

Having gone through all this reasoning, we are now in a position to give our definitive solution to the integral in terms of u :

$$I_u = 4u + 7 \ln|u - 2| + 3 \ln|u + 2| + 5 \ln(u^2 + 4) - 4 \tan^{-1} \frac{1}{2}u + C$$

As in previous cases, this only leaves us with the final step of reversing the substitution with which it all began. We will recall that this substitution was:

$$x = u^4 \text{ so that } u = x^{1/4}$$

Therefore, retaining the fractional-power notation which was present in our original integral, we can very simply now write down:

$$I = 4x^{1/4} + 7 \ln|x^{1/4} - 2| + 3 \ln|x^{1/4} + 2| + 5 \ln|x^{1/2} + 4| - 4 \tan^{-1} \frac{1}{2}x^{1/4} + C$$

Before concluding both the example and the section (perhaps with some sense of relief) we must record a couple of explanatory remarks about the fourth term (that is, the third logarithmic term) in the above solution, where:

$$5 \ln(u^2 + 4) \text{ is transformed into } 5 \ln|x^{1/2} + 4|$$

First, and this is the simpler of the two remarks, we note that u^2 has been transformed into $x^{1/2}$. This is because using the laws of indices, we have the following equivalences:

$$u^2 = (x^{1/4})^2 = x^{2 \cdot (1/4)} = x^{1/2}$$

Second, having replaced u^2 , which we would expect always to remain positive, with $x^{1/2}$, or, as it is better known, $\sqrt{x}$, our guarantee of permanent positive status for the argument of the logarithmic function has evaporated in a puff of smoke, leaving us with the necessity of restoring the modulus brackets to ensure a useful meaning for the term as a whole.

5 Polynomial Division: A Tabular Approach

All of the functions we have integrated in the preceding sections with the aid of a partial fractions decomposition of one sort or another have had polynomial denominators of higher degree than their numerators. Furthermore, the denominators were all susceptible to factorisation to some extent, which is what allowed us to decompose them in such a useful way.

We do, however, encounter expressions which do not fit this pattern, such as the following example:

$$\frac{3x^3 - 5x^2 + 10x - 3}{3x + 1}$$

If asked to integrate this expression, we might find ourselves in some difficulty, as none of our techniques seem suitable to approach the problem.

However, the process known as polynomial division would allow us to restate this expression in the following form:

$$x^2 - 2x + 4 - \frac{7}{3x + 1}$$

We can confirm that this expression is equivalent to the rational expression we started out with by setting out to rewrite it as a single fraction with the denominator $3x + 1$. The numerator of such an expression would be:

$$x^2(3x + 1) - 2x(3x + 1) + 4(3x + 1) - 7$$

This multiplies out to give:

$$3x^3 + x^2 - 6x^2 - 2x + 12x + 4 - 7$$

Combining like terms together produces the final numerator:

$$3x^3 - 5x^2 + 10x - 3$$

which is identical to the numerator of the original expression.

Polynomial division is usually taught as a kind of long division, where we make some sort of guess at each term in the quotient and then confirm its appropriateness and calculate a remainder. I have personally never been fond of long division (my father attempted to teach me how to do it long before I had a grasp of the necessary concepts, and my resulting distaste for a technique which seemed so dependent on guesswork has stayed with me). Because of this distaste, which may well be shared by some of my readers, I intend instead to present here a tabular approach to polynomial division. You may be reminded of the tabular approach to repeated integration by parts which we looked at in [Part 2 Section 6](#) - as with that method, I hope that the tabular approach I will demonstrate here will save time and make the polynomial division process simpler and more mechanical.

The result of a polynomial division operation which I wrote out a little earlier can be integrated fairly easily: the first three terms can be dealt with using the most basic rules of integration, and the final rational term will respond to the rule of integration we have used frequently in the content of our partial fractions examples.

Looking at the second four-term version of our rational expression, we find that the part of this “solution” we are left with after excluding the final rational term is a polynomial of lower degree than the numerator we started with: indeed, its degree is the difference between the degree of the numerator and the degree of the denominator in our original rational expression.

The final term has a constant numerator, and its denominator is the denominator of the expression with which we started out. We could make this statement more general, and say that the degree of the numerator of the final term (which is zero, since it is a constant) is one less than the degree of that term’s denominator (which is 1, since it is a linear expression).

I don’t intend to give a theoretical justification of these observations (although we will find that they hold good in all cases of polynomial division) and, as already stated, I don’t intend to use the “long-division” method to carry out polynomial division in the pages which follow. Instead, I intend to set out a somewhat distinctive method for carrying out these

simplifications, which I will justify for two classes of rational expression where the degree of the numerator is higher than the degree of the denominator: expressions with a linear denominator and expressions with a quadratic denominator. These two cases will suffice to get us through the various examples I intend to present.

The First Class of Expression: Linear Denominators

The example of a rational function reduced to a more easily integrable version above provided us with this equation:

$$\frac{3x^3 - 5x^2 + 10x - 3}{3x + 1} = x^2 - 2x + 4 - \frac{7}{3x + 1}$$

We might represent this in a more general form as follows:

$$\frac{P(x)}{ax + b} = Q(x) + \frac{r}{ax + b}$$

In this general form, $P(x)$ stands for some polynomial in x whose degree is higher than 1 – that is, higher than the degree of the linear denominator underneath it. On the right-hand side, we have another polynomial in x which we can regard as the *quotient* of a polynomial division operation – hence we give it the general name $Q(x)$. The final term comprises a constant *remainder* r over the original denominator $ax + b$.

Let's now see what we can say in general about this equation. First, since both $P(x)$ and $Q(x)$ are polynomials in x , we should be able to set out an expanded form for them. Assuming that $P(x)$ is of an arbitrary degree n , we can express it in a general form as follows:

$$P(x) = p_n x^n + p_{n-1} x^{n-1} + \dots + p_2 x^2 + p_1 x + p_0$$

In the concrete example we have already quoted, we have the following values for the various placeholder variables:

$$n = 3, \text{ so we only have } p_3 = 3, p_2 = -5, p_1 = 10 \text{ and } p_0 = -3$$

In a similar way, we can write out a general expression for the quotient $Q(x)$, which we will recall is of degree $n - 1$, in the following way:

$$Q(x) = q_{n-1} x^{n-1} + q_{n-2} x^{n-2} + \dots + q_2 x^2 + q_1 x + q_0$$

The values taken by the placeholder variables in our concrete example are as follows:

$$n - 1 = 2, \text{ so we have } q_2 = 1, q_1 = -2 \text{ and } q_0 = 4$$

Finally, as for the remainder:

$$r = -7$$

So much for expanded versions of $P(x)$ and $Q(x)$. If we now multiply each side of the above general equation for polynomial division by the linear denominator $ax + b$, we will obtain the following new equation:

$$P(x) = (ax + b)Q(x) + r$$

noting that in the remainder term, the denominator $ax + b$ cancels with the factor we are applying to each term on the right-hand side.

If we restate the quotient $Q(x)$ in the general expanded form we set out above, the first term in the right-hand side can be written as follows:

$$(ax + b)(q_{n-1}x^{n-1} + q_{n-2}x^{n-2} \dots + q_2x^2 + q_1x + q_0)$$

We can distribute the long polynomial expression across the two terms of the opening linear factor to obtain the following two terms:

$$ax(q_{n-1}x^{n-1} + q_{n-2}x^{n-2} \dots + q_2x^2 + q_1x + q_0)$$

and:

$$b(q_{n-1}x^{n-1} + q_{n-2}x^{n-2} \dots + q_2x^2 + q_1x + q_0)$$

The first of these multiplies out to give the following polynomial expression:

$$aq_{n-1}x^n + aq_{n-2}x^{n-1} \dots + aq_2x^3 + aq_1x^2 + aq_0x$$

(See how each of the x terms has its index increased by 1, since we are multiplying through by ax).

In the second case, the polynomial multiples out as follows by the constant b , which makes no contribution to the powers of the x terms:

$$bq_{n-1}x^{n-1} + bq_{n-2}x^{n-2} \dots + bq_2x^2 + bq_1x + bq_0$$

Adding these two polynomials together and gathering like terms now gives us the following coefficients for each power of x (I will set these out in tabular form, since this will make the next stage of our reasoning a little easier):

<i>Power of x</i>	<i>Coefficient</i>
x^n	aq_{n-1}
x^{n-1}	$aq_{n-2} + bq_{n-1}$
...	...
x^2	$aq_1 + bq_2$
x	$aq_0 + bq_1$
1	bq_0

For the most general coefficient corresponding to the power x^i , where $0 < i < n$ - in other words, in all except the first and last rows of the table, we will have a coefficient of the form:

$$aq_{i-1} + bq_i$$

We can now write out the right-hand side of the equation we obtained by multiplying both sides of our general equation of a polynomial division operation as follows:

$$aq_{n-1}x^n + (aq_{n-2} + bq_{n-1})x^{n-1} + \dots + (aq_0 + bq_1)x + bq_0 + r$$

(We have added in the constant remainder term r to make the right-hand side complete.)

Using the general coefficient we derived a little earlier, we might write out our right-hand side using sigma-notation in the following form:

$$aq_{n-1}x^n + \sum_{i=1}^{n-1} (aq_{i-1} + bq_i)x^i + bq_0 + r$$

Having reached this stage, it will be (at the very least) informative to create another table showing the corresponding coefficients on the left-hand and right-hand side of our equation:

$$P(x) = (ax + b)Q(x) + r$$

Bearing in mind our general expansion of $P(x)$, which was as follows:

$$P(x) = p_nx^n + p_{n-1}x^{n-1} + \dots + p_2x^2 + p_1x + p_0$$

this table will be as follows:

Power of x	LHS Coefficient	RHS Coefficient
x^n	p_n	aq_{n-1}
x^{n-1}	p_{n-1}	$aq_{n-2} + bq_{n-1}$
...	...	...
x^2	p_2	$aq_1 + bq_2$
x	p_1	$aq_0 + bq_1$
1	p_0	$bq_0 + r$

Notice that in this table I have added in the remainder r as part of the constant term of the right-hand side polynomial.

The much-used technique of equating coefficients should now allow us to establish a set of equations from this table, one for each row, as follows:

Power of x	Equation of Coefficients
x^n	$aq_{n-1} = p_n$
x^{n-1}	$aq_{n-2} + bq_{n-1} = p_{n-1}$
...	...
x^2	$aq_1 + bq_2 = p_2$
x	$aq_0 + bq_1 = p_1$
1	$bq_0 + r = p_0$

The table can be made more concise if we replace all the rows except the first and last by one single row, which will give a formula from which any of the rows replaced can be obtained:

Power of x	Equation of Coefficients
x^n	$aq_{n-1} = p_n$
x^i where $0 < i < n$	$aq_{i-1} + bq_i = p_i$
1	$bq_0 + r = p_0$

Let's now look back at the rational expression with which we began this discussion:

$$\frac{3x^3 - 5x^2 + 10x - 3}{3x + 1}$$

If we model the way we would like to transform this expression using the general equation for polynomial division which I set out earlier, namely:

$$\frac{P(x)}{ax + b} = Q(x) + \frac{r}{ax + b}$$

We can remark that we know the coefficients of $P(x)$, and we know the values of a and b for the expression we are considering. We don't yet know the coefficients of $Q(x)$, or the value of the remainder r . However, we do have a set of equations which will allow us to relate the unknown items to ones we do know. Let's see where this gets us. First, we record the values of a and b for future reference:

$$a = 3 \text{ and } b = 1$$

The degree n of the polynomial in the numerator is 3, and its coefficients are as follows:

$$p_3 = 3, p_2 = -5, p_1 = 10 \text{ and } p_0 = -3$$

On the basis of the table we prepared above, we can produce an equation for each of these four values, which, when solved, will give us the coefficients of $Q(x)$. Let's look at each equation in turn. Listed together, they are:

$$\begin{aligned} &aq_2 = p_3: 3q_2 = 3 \text{ so that } q_2 = 1 \\ &aq_1 + bq_2 = p_2: 3q_1 + 1 = -5 \text{ so that } 3q_1 = -6 \text{ and } q_1 = -2 \\ &aq_0 + bq_1 = p_1: 3q_0 - 2 = 10 \text{ so that } 3q_0 = 12 \text{ and } q_0 = 4 \\ &bq_0 + r = p_0: 4 + r = -3 \text{ so that } r = -7 \end{aligned}$$

These values give us a right-hand side for our equation of polynomial division which reads as follows:

$$x^2 - 2x + 4 - \frac{7}{3x + 1}$$

exactly the same result as the one I quoted without proof at the start of the discussion.

It will make future calculations of this type more convenient and easy to follow if we set out this working in the form of a table, as follows:

Power of x	Coefficient	Equation Template	Actual Equation	Solution
x^3	$p_3 = 3$	$aq_2 = p_3$	$3q_2 = 3$	$q_2 = 1$
x^2	$p_2 = -5$	$aq_1 + bq_2 = p_2$	$3q_1 + 1 = -5$	$q_1 = -2$
x	$p_1 = 10$	$aq_0 + bq_1 = p_1$	$3q_0 - 2 = 10$	$q_0 = 4$
1	$p_0 = -3$	$bq_0 + r = p_0$	$4 + r = -3$	$r = -7$

The Second Class of Expression: Quadratic Denominators

It will widen the applicability of our tabular approach to polynomial division if we now derive a similar general table of equations which can be used to deal with polynomial division problems where our denominator is a quadratic. Having gone through the previous case for linear denominators in some detail, I intend to deal with the quadratic case in an exclusively symbolic manner.

To start with, then, let's set down our general equation for the kind of polynomial division we are considering here. It will be as follows:

$$\frac{P(x)}{ax^2 + bx + c} = Q(x) + \frac{R(x)}{ax^2 + bx + c}$$

As before, we will allow the numerator on the left-hand side to represent a general polynomial of degree n , as follows:

$$P(x) = p_n x^n + p_{n-1} x^{n-1} + \dots + p_2 x^2 + p_1 x + p_0$$

Since we are dividing by a quadratic, the degree of our quotient will be 2 less than the degree of the numerator (not 1 less, as was the case with a linear denominator). The quotient $Q(x)$ will therefore look like this:

$$Q(x) = q_{n-2} x^{n-2} + q_{n-3} x^{n-3} + \dots + q_2 x^2 + q_1 x + q_0$$

Finally, the remainder term has a numerator which could potentially be a linear expression in x , rather than simply a constant, as previously – because the remainder on division by a quadratic will in general have degree 1 less than the quadratic's degree of 2. We can therefore expand our remainder polynomial $R(x)$ as follows:

$$R(x) = r_1 x + r_0$$

Using a similar method to the one we adopted for the linear denominator case, we will multiply both sides of our general polynomial division equation by $ax^2 + bx + c$, and then equate coefficients in order to obtain a set of equations which, when solved, will allow us to read off the result of the polynomial division effortlessly. After performing this multiplication, we obtain the following equation:

$$P(x) = (ax^2 + bx + c)Q(x) + R(x)$$

Rewriting the first term on the right-hand side using the expanded form of $Q(x)$ which we set out earlier, we get the following:

$$(ax^2 + bx + c)(q_{n-2}x^{n-2} + q_{n-3}x^{n-3} \dots + q_2x^2 + q_1x + q_0)$$

Distributing the long polynomial expansion over the three terms of the opening quadratic factor gives us the following three terms - first:

$$ax^2(q_{n-2}x^{n-2} + q_{n-3}x^{n-3} \dots + q_2x^2 + q_1x + q_0)$$

then:

$$bx(q_{n-2}x^{n-2} + q_{n-3}x^{n-3} \dots + q_2x^2 + q_1x + q_0)$$

and lastly:

$$c(q_{n-2}x^{n-2} + q_{n-3}x^{n-3} \dots + q_2x^2 + q_1x + q_0)$$

The first of these terms multiplies out to give the following result:

$$aq_{n-2}x^n + aq_{n-3}x^{n-1} + \dots + aq_2x^4 + aq_1x^3 + aq_0x^2$$

The second gives:

$$bq_{n-2}x^{n-1} + bq_{n-3}x^{n-2} \dots + bq_2x^3 + bq_1x^2 + bq_0x$$

and from the third one we obtain:

$$cq_{n-2}x^{n-2} + cq_{n-3}x^{n-3} \dots + cq_2x^2 + cq_1x + cq_0$$

Adding these three polynomials together and gathering like terms gives us coefficients for each power of x according to the table below. I have equated each coefficient with the corresponding coefficient from $P(x)$ on the left-hand side of the equation we obtained after our multiplication step earlier. I have also taken account of the “remainder” expression:

$$R(x) = r_1x + r_0:$$

which will make a contribution to the final coefficients of the terms corresponding to the lowest two powers of x - represented here as x and 1 .

Power of x	LHS Coefficient	RHS Coefficient
x^n	p_n	aq_{n-2}
x^{n-1}	p_{n-1}	$aq_{n-3} + bq_{n-2}$
x^{n-2}	p_{n-2}	$aq_{n-4} + bq_{n-3} + cq_{n-2}$
$\dots$	$\dots$	$\dots$
x^2	p_2	$aq_0 + bq_1 + cq_2$
x	p_1	$bq_0 + cq_1 + r_1$
1	p_0	$cq_0 + r_0$

As before, we can create from this a table showing an equation for each coefficient on the left-hand side as follows:

Power of x	LHS Coefficient	Equation of Coefficients
x^n	p_n	$aq_{n-2} = p_n$
x^{n-1}	p_{n-1}	$aq_{n-3} + bq_{n-2} = p_{n-1}$
x^{n-2}	p_{n-2}	$aq_{n-4} + bq_{n-3} + cq_{n-2} = p_{n-2}$
...	...	...
x^2	p_2	$aq_0 + bq_1 + cq_2 = p_2$
x	p_1	$bq_0 + cq_1 + r_1 = p_1$
1	p_0	$cq_0 + r_0 = p_0$

Just as before, we can now replace all rows except for the first two and the last two with one single row, giving a formula which can be used to represent all rows except for the four which are given explicitly – these are the first two and the last two:

Power of x	Equation of Coefficients
x^n	$aq_{n-2} = p_n$
x^{n-1}	$aq_{n-3} + bq_{n-2} = p_{n-1}$
x^i where $1 < i < n-1$	$aq_{i-2} + bq_{i-1} + cq_i = p_i$
x	$bq_0 + cq_1 + r_1 = p_1$
1	$cq_0 + r_0 = p_0$

It's important to see how this table lets us perform polynomial division by a quadratic easily by showing an example. We will try evaluating this rational expression:

$$\frac{x^7 + 5x^6 + 10x^5 + 14x^4 + 18x^3 + 25x^2 + 25x + 11}{x^2 + 2x + 2}$$

This expression comprises a numerator of degree 7 divided by a quadratic. This should give us enough coefficients in the numerator to exercise our tabular method fully.

Just as we did when we were working with a linear denominator, we will represent the algebraic rearrangement we wish to make with our general equation for division by a quadratic:

$$\frac{P(x)}{ax^2 + bx + c} = Q(x) + \frac{R(x)}{ax^2 + bx + c}$$

In the present concrete example, we have the following values for the coefficients a , b and c :

$$a = 1, b = 2 \text{ and } c = 2$$

Sticking with the notation we have been using for the coefficients of $P(x)$, our current example gives us the following values:

$$p_7 = 1, p_6 = 5, p_5 = 10, p_4 = 14, p_3 = 18, p_2 = 25, p_1 = 25 \text{ and } p_0 = 11$$

The number of coefficients provided here (in other words, the degree of the numerator) defines the extent of the table which will give us our solution – ie $n = 7$. We might note before we start that the two “special case” equations at the top of our general table template will apply where the powers of x in the left-hand column are 7 and 6; the two “special cases” at the bottom of the table template will apply when the powers of x are 1 and 0. For the other powers of x - those from 5 down to 2 inclusive, we will use the “general” equation with the i subscripts. The table can now be written down easily using the above values, and is as follows:

Power of x	Coefficient	Equation Template	Actual Equation	Solution
x^7	$p_7 = 1$	$aq_5 = p_7$	$q_5 = 1$	$q_5 = 1$
x^6	$p_6 = 5$	$aq_4 + bq_5 = p_6$	$q_4 + 2 = 5$	$q_4 = 3$
x^5	$p_5 = 10$	$aq_3 + bq_4 + cq_5 = p_5$	$q_3 + 8 = 10$	$q_3 = 2$
x^4	$p_4 = 14$	$aq_2 + bq_3 + cq_4 = p_4$	$q_2 + 10 = 14$	$q_2 = 4$
x^3	$p_3 = 18$	$aq_1 + bq_2 + cq_3 = p_3$	$q_1 + 12 = 18$	$q_1 = 6$
x^2	$p_2 = 25$	$aq_0 + bq_1 + cq_2 = p_2$	$q_0 + 20 = 25$	$q_0 = 5$
x	$p_1 = 25$	$bq_0 + cq_1 + r_1 = p_1$	$22 + r_1 = 25$	$r_1 = 3$
1	$p_0 = 11$	$cq_0 + r_0 = p_0$	$10 + r_0 = 11$	$r_0 = 1$

If we now write out the right-hand side of our general equation for polynomial division by a quadratic, adhering to our conventions regarding the names of the various coefficients, we will arrive at the following solution to our polynomial division problem:

$$x^5 + 3x^4 + 2x^3 + 4x^2 + 6x + 5 + \frac{3x + 1}{x^2 + 2x + 2}$$

A Note on the Process of Division by a Quadratic

I remarked in passing during the preceding discussion that, in carrying out polynomial division using the tabular method, the number of rows in the table we use to create and read off the result is determined by the degree of the polynomial in the numerator. This is represented as $P(x)$ in my discussion above.

For a numerator of degree n , we would expect the number of rows to be $n + 1$. This is because there will be one row corresponding to the coefficient of each of the powers of x in the numerator, and for a numerator of degree n , this will amount to one row for each explicit power of x from n down to 1, as well as one further row for the constant term in the polynomial (which, of course, corresponds to the “zeroth” power of x , ie x^0 , which equals 1). Thus we obtain a total of $n + 1$ rows in our table.

We ought to note in addition, however, that we might expect the table for this kind of division to have at the very least four lines, in order to accommodate each of the four “special case” rows we see in the general table template. It therefore makes sense for us to take a moment to consider what happens in the case of a numerator of the smallest appropriate degree. Such a numerator would be a cubic polynomial, since this exemplifies the lowest degree of polynomial we would expect to be required to divide by a quadratic.

Our reasoning a couple of paragraphs ago would predict that such a case would produce a table with 4 rows (this is $n + 1$ where $n = 3$, the degree of a cubic). This would happily

accommodate all our four “special case” table lines, leaving the “general” case unused in this instance. It will be instructive to take a quick look at an example of this type.

We will accordingly perform polynomial division on the following rational expression:

$$\frac{3x^3 - 2x^2 + 4x - 3}{x^2 + 3x + 3}$$

We can see that, in our usual notation, the coefficients of the divisor are as follows:

$$a = 1, b = 3 \text{ and } c = 3$$

There is probably no need if you have stayed with me up to this point to list the coefficients of the numerator polynomial separately, since these will be clearly set out in our table. This table will be as follows:

<i>Power of x</i>	<i>Coefficient</i>	<i>Equation Template</i>	<i>Actual Equation</i>	<i>Solution</i>
x^3	$p_3 = 3$	$aq_1 = p_3$	$q_1 = 3$	$q_1 = 3$
x^2	$p_2 = -2$	$aq_0 + bq_1 = p_2$	$q_0 + 9 = -2$	$q_0 = -11$
x	$p_1 = 4$	$bq_0 + cq_1 + r_1 = p_1$	$r_1 - 24 = 4$	$r_1 = 28$
1	$p_0 = -3$	$cq_0 + r_0 = p_0$	$r_0 - 33 = -3$	$r_0 = 30$

I must point out for the sake of clarity that I have rearranged the equations of coefficients in the bottom two lines of the table to accommodate my distaste for opening minus signs. By way of example, I have restated the final line’s equation as follows:

$$-33 + r_0 = -3 \text{ is restated as } r_0 - 33 = -3$$

We can immediately read off our result as follows:

$$3x - 11 + \frac{28x + 30}{x^2 + 3x + 3}$$

We can see that this table actually produces four constants: q_1, q_0, r_1 and r_0 . It is reasonable to expect that this would need four equations, especially since in the tabular technique I have proposed here, each equation of coefficients except the one in the top row uses information from preceding rows to obtain a new constant. The four constants we obtain represent the two coefficients of the quotient $Q(x)$, which is a linear expression (being, as it is, the quotient of a degree-3 polynomial and a degree-2 polynomial: and therefore itself of degree 1), as well as the two coefficients of the remainder expression $R(x)$, which is once again a linear expression.

We will now proceed to a number of examples of the use of polynomial division as a tool to aid the process of integration of rational functions, which is, after all, the whole point of discussing the technique at such length!

Example 1

We will begin by integrating the rational expression we used to illustrate the tabular approach to division by a linear denominator. I do this because the actual work of polynomial division

was done earlier in this section, so that we can concentrate here on the actual process of integration which follows on from the algebraic rearrangement of the integrand.

Our integral here will therefore be:

$$I = \int \frac{3x^3 - 5x^2 + 10x - 3}{3x + 1} dx$$

The polynomial division we performed earlier gives us the following equivalent integral:

$$\int \left(x^2 - 2x + 4 - \frac{7}{3x + 1} \right) dx$$

We have all the technical tools necessary to integrate this easily term by term: we will use the standard rule for the integration of simple powers of x , as well as the rule which we applied repeatedly in the sections above dealing with partial fractions. The result which emerges is very quickly written down as follows:

$$I = \frac{1}{3}x^3 - x^2 + 4x - \frac{7}{3}\ln|3x + 1| + C$$

Example 2

Continuing with our determination to complete unfinished business, we will now proceed to integrate the rational expression we used to illustrate tabular division by a quadratic. Our integral here is the rather fearsome one below:

$$I = \int \frac{x^7 + 5x^6 + 10x^5 + 14x^4 + 18x^3 + 25x^2 + 25x + 11}{x^2 + 2x + 2} dx$$

Polynomial division reduces this to the much less appalling form:

$$\int \left(x^5 + 3x^4 + 2x^3 + 4x^2 + 6x + 5 + \frac{3x + 1}{x^2 + 2x + 2} \right) dx$$

Once again, let's start integrating this term by term. We get as far as this before we have to give some thought to our next step:

$$\frac{1}{6}x^6 + \frac{3}{5}x^5 + \frac{1}{2}x^4 + \frac{4}{3}x^3 + 3x^2 + 5x + \int \frac{3x + 1}{x^2 + 2x + 2} dx$$

Let's identify the remaining integral at the end of this expression as follows for ease of reference:

$$I_1 = \int \frac{3x + 1}{x^2 + 2x + 2} dx$$

A quick look at the end of [Part 1](#) of this book will confirm that this is an integral of the form:

$$\int \frac{Ax + B}{Cx^2 + Dx + E} dx$$

We integrated an example of this type in [Part 1 Section 11 Example 2](#), and we will use the same technique – a little abbreviated, since we will be applying the technique rather than giving a full explanation. (This was the purpose of [Part 1 Section 11](#)!)

We will begin by completing the square for the quadratic denominator of the integrand in I_1 . This goes as follows:

$$x^2 + 2x + 2 = (x + 1)^2 + 1$$

and it means that we can rewrite I_1 thus:

$$\int \frac{3x + 1}{(x + 1)^2 + 1} dx$$

Our next step is to make the substitution:

$$u = x + 1$$

This has as consequence the following:

$$\frac{du}{dx} = 1 \text{ so that } dx = du$$

Furthermore, by rearranging the substitution formula, we get:

$$x = u - 1 \text{ so that } 3x + 1 = 3u - 2$$

Putting all these observations into the integrand of I_1 , we now have the following integral in terms of u :

$$I_u = \int \frac{3u - 2}{u^2 + 1} du$$

We can split this into two separate integrals, which we will give specific names, as follows:

$$I_2 = \int \frac{3u}{u^2 + 1} du \text{ and } I_3 = \int \frac{2}{u^2 + 1} du \text{ so that } I_u = I_2 - I_3$$

We now notice that I_2 contains a function-derivative pair, and is of the classic pattern for straightforward substitution. We have already evaluated this integral (although without the coefficient of 3 which is present here) in [Part 1 Section 8](#), and it is listed as a useful standard integral in [Appendix 4 Section 2](#). Its solution is as follows – making sure to restore the coefficient 3:

$$I_2 = \frac{3}{2} \ln(u^2 + 1) + C$$

With similar ease, we note that I_3 has the form of a version of the standard integral which evaluates to an inverse tangent function, as mentioned as well in [Appendix 4 Section 2](#). The solution here, with the added coefficient of 2, is:

$$I_3 = 2 \tan^{-1} u + C$$

Recalling the formula we gave earlier for I_u in terms of its two components, we can now write down its solution in terms of u as follows:

$$I_u = I_2 - I_3 \text{ so that } I_u = \frac{3}{2} \ln(u^2 + 1) - 2 \tan^{-1} u + C$$

Reversing the original u -substitution, we now have the following solution in terms of x :

$$\frac{3}{2} \ln((x+1)^2 + 1) - 2 \tan^{-1}(x+1) + C$$

To get this into a better final form, we just have to notice that the argument of the logarithmic function is the completed-square version of the denominator of our integrand, and we can therefore rewrite the solution as follows:

$$I_1 = \frac{3}{2} \ln(x^2 + 2x + 2) - 2 \tan^{-1}(x+1) + C$$

The final solution to our original integral I can therefore be written down as follows:

$$I = \frac{1}{6}x^6 + \frac{3}{5}x^5 + \frac{1}{2}x^4 + \frac{4}{3}x^3 + 3x^2 + 5x + \frac{3}{2} \ln(x^2 + 2x + 2) - 2 \tan^{-1}(x+1) + C$$

In the next two examples, we will go through the entire process of evaluation, carrying out the required polynomial division and then completing the integration operation.

Example 3

Our integral in this example will be as follows:

$$I = \int \frac{x^3 - 1}{x + 2} dx$$

The integrand is quite a compact rational expression, compared to some of the monstrosities we have been looking at so far, but it does nonetheless constitute a cubic polynomial divided by a linear denominator. As such, it is clearly a candidate for the use of polynomial division in order to transform it into an expression which can be integrated using the rules we have at our disposal. We should just pause to remark that a number of the coefficients of powers of x in the model we have adopted for the numerator polynomial will here turn out to be zero.

Without further preamble, we can get the polynomial division step out of the way very quickly by using the table template for division by a linear denominator. We must, of course, record the important fact that if we model out denominator by our standard general expression $ax + b$, we have the following values for our constants:

$$a = 1 \text{ and } b = 2$$

The table for the current division problem is therefore as follows:

Power of x	Coefficient	Equation Template	Actual Equation	Solution
x^3	$p_3 = 1$	$aq_2 = p_3$	$q_2 = 1$	$q_2 = 1$
x^2	$p_2 = 0$	$aq_1 + bq_2 = p_2$	$q_1 + 2 = 0$	$q_1 = -2$
x	$p_1 = 0$	$aq_0 + bq_1 = p_1$	$q_0 - 4 = 0$	$q_0 = 4$
1	$p_0 = -1$	$bq_0 + r = p_0$	$8 + r = -1$	$r = -9$

From the table, reading off the results in the right-hand column, we can rewrite our integrand as follows:

$$\frac{x^3 - 1}{x + 2} = x^2 - 2x + 4 - \frac{9}{x + 2}$$

Our integral has now become the following much easier problem:

$$\int \left(x^2 - 2x + 4 - \frac{9}{x + 2} \right) dx$$

and we can evaluate this using the rule for the integration of powers of x and the rule for simple rational expressions with constant numerator and linear denominator which we have so often used in the preceding sections. The solution is:

$$I = \frac{1}{3}x^3 - x^2 + 4x - 9 \ln|x + 2| + C$$

Example 4

For this example, we will progress to a quadratic denominator, and an integrand which contains more of the components we have become used to. Our integral will be the following:

$$I = \int \frac{x^3 - 5x^2 + 3x - 15}{x^2 + 3} dx$$

Let's proceed straight to the polynomial division step, bearing in mind that using our standard model of $ax^2 + bx + c$ for the quadratic denominator, the constants will be as follows:

$$a = 1, b = 0 \text{ and } c = 3$$

The table which will allow us to read off the results of the division process will then be as follows:

Power of x	Coefficient	Equation Template	Actual Equation	Solution
x^3	$p_3 = 1$	$aq_1 = p_3$	$q_1 = 1$	$q_1 = 1$
x^2	$p_2 = -5$	$aq_0 + bq_1 = p_2$	$q_0 = -5$	$q_0 = -5$

x	$p_1 = 3$	$bq_0 + cq_1 + r_1 = p_1$	$3 + r_1 = 3$	$r_1 = 0$
1	$p_0 = -15$	$cq_0 + r_0 = p_0$	$r_0 - 15 = -15$	$r_0 = 0$

This is a rather unusual result, as compared to everything which has gone before – both the coefficients of the linear remainder term are zero! What this means is that the restatement of our rational integrand looks like this:

$$\frac{x^3 - 5x^2 + 3x - 15}{x^2 + 3} = x - 5 + \frac{0}{x^2 + 3}$$

In a more concise form, this can be written:

$$\frac{x^3 - 5x^2 + 3x - 15}{x^2 + 3} = x - 5$$

So we see that the cubic numerator divides *exactly* by the quadratic denominator to give a linear factor without remainder. In other words, the entire rational expression can be simplified to a very simple linear expression. We have actually managed to *factorise* the numerator into a quadratic and a linear factor – multiplying both sides of the above equation by $x^2 + 3$ gives us the following equivalence:

$$x^3 - 5x^2 + 3x - 15 = (x^2 + 3)(x - 5)$$

However, to return to the task in hand: our integral is now reduced to a very simple one:

$$\int (x - 5)dx$$

This integrates almost without effort to give the following result:

$$I = \frac{1}{2}x^2 - 5x + C$$

Despite the impression that what we have here is a somewhat trivial result compared with our other examples, I have included it in order to be able to make a few useful observations.

First, it is clearly not impossible that you may encounter rational expressions calling for polynomial division which produce results having no remainder (just as dividing some numbers by others can produce no remainder). It is important to be aware of this, and to be equally aware that our tabular method of polynomial division still works in these cases, and demonstrates them to us by producing zero coefficients for the remainder term.

It also hints at the possibility of using the tabular polynomial division technique for factorising polynomials, especially where (as in the current example) the first and last coefficients of the denominator exactly divide those of the numerator.

Example 5

For our final example in this section, we will look at another case of polynomial division by a quadratic. Our integral will be as follows:

$$I = \int \frac{x^4 + 4}{x^2 - 5} dx$$

Like [Example 3](#) earlier, this is quite a concise expression, and some of our constants and coefficients will turn out to be zero. This will serve to make the process of completing the table which will give the result of the division operation a little easier, but the tabular approach will be needed nonetheless.

Let's first set down the values of the coefficients in the quadratic denominator, using our standard model:

$$a = 1, b = 0 \text{ and } c = -5$$

We can now write out our table for the division operation as follows:

<i>Power of x</i>	<i>Coefficient</i>	<i>Equation Template</i>	<i>Actual Equation</i>	<i>Solution</i>
x^4	$p_4 = 1$	$aq_2 = p_4$	$q_2 = 1$	$q_2 = 1$
x^3	$p_3 = 0$	$aq_1 + bq_2 = p_3$	$q_1 = 0$	$q_1 = 0$
x^2	$p_2 = 0$	$aq_0 + bq_1 + cq_2 = p_2$	$q_0 - 5 = 0$	$q_0 = 5$
x	$p_1 = 0$	$bq_0 + cq_1 + r_1 = p_1$	$r_1 = 0$	$r_1 = 0$
1	$p_0 = 4$	$cq_0 + r_0 = p_0$	$r_0 - 25 = 4$	$r_0 = 29$

As usual, we can now read off our solution from the right-hand column; it turns out to be as follows:

$$x^2 + 5 + \frac{29}{x^2 - 5}$$

A couple of observations may be of interest: first, the remainder expression, which is the numerator of the concluding rational expression, is only a numeric constant in this case, rather than the full-blown linear expression we have sometimes obtained (since the coefficient r_1 which multiplies x in the remainder expression was found to be zero); and second, the quotient expression $x^2 + 5$ only differs from the denominator of the original rational expression by a change of sign.

Let's carry straight on to the actual job of integration. Having performed the division operation, we can now restate our original integral as follows:

$$\int \left(x^2 + 5 + \frac{29}{x^2 - 5} \right) dx$$

The first two terms integrate easily; our integral now becomes:

$$\frac{1}{3}x^3 + 5x + \int \frac{29}{x^2 - 5} dx$$

We will name the remaining integral for convenience as follows:

$$I_1 = \int \frac{29}{x^2 - 5} dx, \text{ or } 29 \int \frac{1}{x^2 - 5} dx$$

While this integral may look a little baffling at first (the minus sign in the denominator rules it out as the integral of an inverse tangent expression, and we can't complete the square for the denominator, since it is already in the form which would result from such a procedure). We can, however, call to our aid one of the standard integrals we obtained using partial fractions and quoted in [Section 1](#) above. The standard integral in question is the following one:

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + C$$

In the case of our current integral I_1 , we note that $a^2 = 5$, so that $a = \sqrt{5}$. We can then use the standard integral just quoted to obtain the following solution for I_1 :

$$I_1 = \frac{29}{2\sqrt{5}} \ln \left| \frac{x - \sqrt{5}}{x + \sqrt{5}} \right| + C$$

And we can plug this straight back into our last draft for the solution to our original integral I to obtain this solution:

$$I = \frac{1}{3}x^3 + 5x + \frac{29}{2\sqrt{5}} \ln \left| \frac{x - \sqrt{5}}{x + \sqrt{5}} \right| + C$$

I have used a standard integral derived using the Method of Partial Fractions to complete this integration problem. Those who would like to see the integration of I_1 carried out by hand may consult [Appendix 5 Section 18](#), where this process is presented for reference.

Having made use of a standard integral to deal with the remainder term after division by a quadratic in this example, it will no doubt have occurred to you that it might have been useful to do the same with the remainder term in [Example 2](#) above, where we had to evaluate an integral of the form:

$$\int \frac{Ax + B}{Cx^2 + Dx + E} dx$$

We did this by substitution followed by splitting the resulting integral in terms of u into two components. These components turned out to be instances of standard integrals we have already observed in this book. I have derived a symbolic solution for integrals of the above kind where $C = 1$ (in other words, where the denominator is a monic quadratic, as it was in [Example 2](#) above) in [Appendix 5 Section 19](#), and the resulting standard integral is listed in [Appendix 4 Section 2](#) for easy reference.

It should be clearly understood however, that the formula derived in [Appendix 5](#) only applies to cases where the denominator is an *irreducible* quadratic – that is, one with no real roots, and which cannot be factorised into two linear factors. If our denominator can be factorised in this way – as we found in the current example – then the integration of our rational expression becomes a partial fractions problem, and should be dealt with accordingly. In the

current example, the remainder term was evaluated using a standard integral which was derived from consideration of the partial fractions technique.

6 Power Substitution and Polynomial Division

In [Section 4](#) above, we examined some integration problems where the use of power substitution led to an integral in terms of the substitution variable u which required partial fractions for its solution. In a similar way, we will now consider some examples of rational functions involving roots which, after power substitution, turn into rational functions in terms of u which call for polynomial division in order to complete their integration.

Broadly speaking, it is the relationship between the numerator and denominator of a rational function which directs us to rearrange it using either partial fractions (if this is indeed possible) or polynomial division in order to obtain a result which can be integrated using the tools at our disposal. In particular, a rational expression where the denominator is of lower degree as a polynomial than the numerator will be amenable to polynomial division.

Example 1

We will begin by considering the following integral:

$$I = \int \frac{2 + \sqrt{x}}{3 - \sqrt{x}} dx$$

In accordance with our well-established practice when dealing with integrands which contain roots, we will make a power substitution in order to take the roots (which are both square roots in this case) out of consideration. We will substitute:

$$u^2 = x \text{ so that } \sqrt{x} = \sqrt{u^2} = u$$

Differentiating x in terms of u then gives us:

$$\frac{dx}{du} = 2u \text{ so that } dx = 2u du$$

We can use these facts to obtain the following integral in terms of u :

$$\int \frac{2 + u}{3 - u} \cdot 2u du = \int \frac{2u^2 + 4u}{3 - u} du$$

I have changed the order of terms in the numerator when multiplying by $2u$ in order to put it into a more traditional shape for the working which follows. We will need to do the same with the denominator, which I have left as it was here to accommodate my distaste for opening minus signs. We might note that it could easily be rewritten as follows:

$$3 - u = -u + 3$$

With this possible reordering in mind, we can see that we now have as our integrand a rational function in terms of u which has a quadratic numerator and a linear denominator – a

clear candidate for polynomial division by a linear factor. Modelling this rational expression and the result we will obtain after polynomial division in our standard way as follows:

$$\frac{p_2u^2 + p_1u + p_0}{au + b} = q_1u + q_0 + \frac{r}{au + b}$$

we can start by writing down the values for a and b as follows:

$$a = -1 \text{ and } b = 3$$

The actual work of polynomial division is now carried out with the following table:

Power of u	Coefficient)	Equation Template	Actual Equation	Solution
u^2	$p_2 = 2$	$aq_1 = p_2$	$-q_1 = 2$	$q_1 = -2$
u	$p_1 = 4$	$aq_0 + bq_1 = p_1$	$-q_0 - 6 = 4$	$q_0 = -10$
1	$p_0 = 0$	$bq_0 + r = p_0$	$r - 30 = 0$	$r = 30$

Putting the values in the right-hand column into the right-hand side of the model we set up a little earlier, we have the following equation:

$$\frac{2u^2 + 4u}{3 - u} = -2u - 10 + \frac{30}{3 - u}$$

Our integral in terms of u is now in the following more easily handled form:

$$\int \left(-2u - 10 + \frac{30}{3 - u} \right) du$$

which can be integrated term by term. In order to make it easier to use our standard integration rule on the final term, I will rearrange its denominator using the following standard algebraic equivalence:

$$a - b = -(b - a)$$

This will require us to change the plus sign in front of the final term to a minus sign, so that our integral is now in the following form, which we can easily deal with term by term:

$$\int \left(-2u - 10 - \frac{30}{u - 3} \right) du$$

Its solution in terms of u is as follows:

$$-u^2 - 10u - 30 \ln|u - 3| + C$$

and we can obtain our final solution by reversing the substitution as follows:

$$I = -x - 10\sqrt{x} - 30 \ln|\sqrt{x} - 3| + C$$

(Alas, nothing really clever can be done about the minus signs which seem to suffuse this expression.)

Example 2

Let's proceed to an integral involving a cube root; we will consider this one:

$$I = \int \frac{3}{4 + x^{1/3}} dx$$

As before, the first step will be to address the cube root by means of a power substitution, as follows:

$$u^3 = x \text{ so that } x^{1/3} = (u^3)^{1/3} = u$$

Differentiating x in terms of u will give us:

$$\frac{dx}{du} = 3u^2 \text{ so that } dx = 3u^2 du$$

These facts allow us to transform our integral in x into the following integral in terms of u :

$$\int \frac{3}{4 + u} \cdot 3u^2 du = \int \frac{9u^2}{u + 4} du$$

Once again, we have a quadratic numerator divided by a linear denominator – admittedly, a much attenuated quadratic with all but one coefficient equal to zero. However, this need not deter us. Using the same model for the polynomial division process which we used in [Example 1](#), we can set down the following facts:

$$a = 1 \text{ and } b = 4$$

The polynomial division table is as follows:

Power of u	Coefficient	Equation Template	Actual Equation	Solution
u^2	$p_2 = 9$	$aq_1 = p_2$	$q_1 = 9$	$q_1 = 9$
u	$p_1 = 0$	$aq_0 + bq_1 = p_1$	$q_0 + 36 = 0$	$q_0 = -36$
1	$p_0 = 0$	$bq_0 + r = p_0$	$r - 144 = 0$	$r = 144$

We can now read off the results from the right-hand column in order to obtain the following equation for our integrand in terms of u :

$$\frac{9u^2}{u + 4} = 9u - 36 + \frac{144}{u + 4}$$

We now only need to evaluate the following integral in terms of u :

$$\int \left(9u - 36 + \frac{144}{u + 4} \right) du$$

This works out easily term by term to give this solution:

$$\frac{9}{2}u^2 - 36u + 144 \ln|u + 4| + C$$

The final step of reversing the substitution gives us the following solution in terms of x :

$$I = \frac{9}{2}x^{2/3} - 36x^{1/3} + 144 \ln|x^{1/3} + 4| + C$$

Example 3

Our last example in this section will be an integral where we are required to deal with multiple roots of differing kinds. It will be as follows:

$$I = \int \frac{\sqrt{x} + 1}{\sqrt{x}(\sqrt[3]{x} + 1)} dx$$

Here we find two square roots and a cube root. We must therefore choose a power substitution which will mop up all of these. Following the approach we took in [Part 1 Section 9](#), we will choose the following substitution:

$$u^6 = x \text{ so that } \sqrt{x} = (u^6)^{1/2} = u^3 \text{ and } \sqrt[3]{x} = (u^6)^{1/3} = u^2$$

(The reason for our choice is based on the principle we established in [Part 1 Section 9 Example 4](#), where we noted that the choice of power of u in our substitution expression to deal with the particular roots we find in the integrand should be the lowest common multiple of 2 and 3.)

Differentiating x in terms of u will give us:

$$\frac{dx}{du} = 6u^5 \text{ so that } dx = 6u^5 du$$

Let's now put all this back into our integral, in order to obtain the following integral in terms of u :

$$\int \frac{u^3 + 1}{u^3(u^2 + 1)} \cdot 6u^5 du = \int \frac{u^3 + 1}{u^2 + 1} \cdot 6u^2 du = \int \frac{6u^5 + 6u^2}{u^2 + 1} du$$

This is the first occasion in this section when we have seen the mounting up of powers of u which is typical of many integrations using power substitution. In order to make our next step a little easier, we will factorise out the common coefficient 6 in the numerator, so that our integral in u becomes:

$$6 \int \frac{u^5 + u^2}{u^2 + 1} du$$

We can therefore express I_u , our integral in terms of u , using the following abbreviation:

$$I_u = 6I_1 \text{ where } I_1 = \int \frac{u^5 + u^2}{u^2 + 1} du$$

and our task is now reduced to the evaluation of I_1 . We will begin by performing polynomial division on the integrand, to get it into a form which we can more easily integrate. Here, of course, we have a quadratic denominator and a quintic, or degree-5, polynomial in the numerator.

We can therefore model out polynomial division operation as follows:

$$\frac{p_5u^5 + p_4u^4 + p_3u^3 + p_2u^2 + p_1u + p_0}{au^2 + bu + c} = q_3u^3 + q_2u^2 + q_1u + q_0 + \frac{r_1u + r_0}{au^2 + bu + c}$$

noting, of course, that a good many of the numerator coefficients p_i will be zero.

We can record the coefficients of our denominator as follows:

$$a = 1, b = 0 \text{ and } c = 1$$

and our polynomial division table will look like this:

Power of u	Coefficient	Equation Template	Actual Equation	Solution
u^5	$p_5 = 1$	$aq_3 = p_5$	$q_3 = 1$	$q_3 = 1$
u^4	$p_4 = 0$	$aq_2 + bq_3 = p_4$	$q_2 = 0$	$q_2 = 0$
u^3	$p_3 = 0$	$aq_1 + bq_2 + cq_3 = p_3$	$q_1 + 1 = 0$	$q_1 = -1$
u^2	$p_2 = 1$	$aq_0 + bq_1 + cq_2 = p_2$	$q_0 = 1$	$q_0 = 1$
u	$p_1 = 0$	$bq_0 + cq_1 + r_1 = p_1$	$r_1 - 1 = 0$	$r_1 = 1$
1	$p_0 = 0$	$cq_0 + r_0 = p_0$	$1 + r_0 = 0$	$r_0 = -1$

We can now read off the result of our polynomial division from the fourth column, to obtain the following:

$$u^3 - u + 1 + \frac{u - 1}{u^2 + 1}$$

which means that we can now rewrite our intermediate integral I_1 as follows:

$$\int \left(u^3 - u + 1 + \frac{u - 1}{u^2 + 1} \right) du = \int \left(u^3 - u + 1 + \frac{u}{u^2 + 1} - \frac{1}{u^2 + 1} \right) du$$

In the second version above, I have split the rational expression in the final term into two components, which we can integrate without much more ado. The first of these two parts is an instance of the standard integral:

$$\int \frac{x}{x^2 + 1} dx = \frac{1}{2} \ln(x^2 + 1) + C$$

evaluated in [Part 1 Section 7](#) and listed in [Appendix 4 Section 2](#). The second part is an instance of the standard integral:

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{1}{a} x + C$$

with $a = 1$. This standard integral is, of course, commonly encountered in the following form, which suits our case:

$$\int \frac{1}{x^2 + 1} dx = \tan^{-1} x + C$$

These observations let us write down the solution to I_1 as follows:

$$I_1 = \frac{1}{4}u^4 - \frac{1}{2}u^2 + u + \frac{1}{2}\ln(u^2 + 1) - \tan^{-1} u + C$$

Recalling our earlier formula which allows us to derive I_u from the intermediate integral I_1 - this formula was:

$$I_u = 6I_1$$

we can now write down our complete solution in terms of u :

$$I_u = \frac{3}{2}u^4 - 3u^2 + 6u + 3\ln(u^2 + 1) - 6\tan^{-1} u + C$$

And the last step of reversing the substitution yields our final solution. In order to complete it, we must note that our substitution:

$$u^6 = x \text{ implies that } u = x^{1/6}$$

This conclusion, combined with several applications of the laws of indices, brings us to the following end result:

$$I = \frac{3}{2}x^{2/3} - 3x^{1/3} + 6x^{1/6} + 3\ln|x^{1/3} + 1| - 6\tan^{-1} x^{1/6} + C$$

We note here that the solution to our integration problem is found to contain two instances of the sixth root of x , although our original integrand only contained square and cube roots.

7 Irrational Functions

In this section, we will look at some specific examples of the integration of irrational functions of a kind we have not so far encountered. It will be seen that power substitution is an important tool in integrating these functions, although other techniques and tools, including the use of algebra, will also come into play. It is because of the interplay of techniques which we will apply that I have devoted a separate section to these examples.

Example 1

We will start off gently, by performing the following integration:

$$I = \int \frac{1}{x} \sqrt{\frac{1-x}{x}} dx$$

Pausing briefly to remark that we might be tempted to try integration by parts to deal with what is certainly a product of two functions in x , I will instead start by allowing the presence of a complicated square root expression to sway me towards power substitution.

Following the pattern of previous power substitutions, where we choose our substitution so as to remove the square root from consideration, we will try substituting:

$$u^2 = \frac{1-x}{x} \text{ so that } \sqrt{\frac{1-x}{x}} = \sqrt{u^2} = u$$

Before carrying on any further, let's pause to see what the implications of our substitution choice will be for dealing with the factor of $1/x$ which comes immediately before the square root in our integrand. We can split the right-hand side of our substitution formula into two separate fractional expressions to get this:

$$u^2 = \frac{1-x}{x} = \frac{1}{x} - \frac{x}{x} = \frac{1}{x} - 1 \text{ so that } \frac{1}{x} = u^2 + 1$$

This might be regarded as a bit of a lucky break, as it arises directly out of the choice I made for the expression under the square root sign. However, to proceed – the last equation above has this as its consequence:

$$x = \frac{1}{u^2 + 1}$$

which makes it somewhat easier to find an expression for dx , since differentiating x with respect to u is a bit easier than it would be to differentiate u in terms of x using the expression for u which we derived above. We will differentiate using the Quotient Rule, and this gives us:

$$\frac{dx}{du} = \frac{0 \cdot (u^2 + 1) - 1 \cdot 2u}{(u^2 + 1)^2} = -\frac{2u}{(u^2 + 1)^2} \text{ so that } dx = -\frac{2u}{(u^2 + 1)^2} du$$

We can now plug all this into our original integral to get the following integral in terms of u :

$$\int (u^2 + 1) \cdot u \cdot \left(-\frac{2u}{(u^2 + 1)^2} \right) du = -2 \int \frac{u^2}{u^2 + 1} du$$

We know how to integrate:

$$\int \frac{x^2}{x^2 + 1} dx$$

since it has been integrated in two different ways elsewhere in the text, as explained in some notes accompanying its listing as a standard integral in [Appendix 4 Section 2](#). The solution of the integral in terms of u above (without the coefficient of -2 outside the integral sign) is therefore as follows:

$$u - \tan^{-1} u + C$$

Restoring the coefficient of -2 - and noting that its negative sign can be used to reverse the order of the terms in the above expression, we have the following solution in terms of u :

$$2\tan^{-1} u - 2u + C$$

We can now round things off by reversing the substitution using the expression for u in terms of x which we derived earlier – this expression was:

$$u = \sqrt{\frac{1-x}{x}}$$

and it gives us the following final solution in x :

$$2\tan^{-1} \sqrt{\frac{1-x}{x}} - 2\sqrt{\frac{1-x}{x}} + C$$

As a closing remark, we might recall that I confessed to being tempted to use integration by parts for this example. The solution we have obtained here looks very like the result of an application of this technique; this should not be altogether surprising, since the standard integral we used towards the end of our working can indeed be derived using integration by parts, as noted in [Appendix 4 Section 2](#).

Example 2

This time, we will move into what may at first seem like unknown territory; it will turn out not to be so unknown after all, however, if we stick to tried and tested principles. Our integral will be:

$$I = \int \sqrt{e^x + 1} dx$$

Once again, the square root sign, which encloses the whole of the integrand this time, points us towards power substitution. Here, we will substitute:

$$u^2 = e^x + 1 \text{ so that } \sqrt{e^x + 1} = \sqrt{u^2} = u$$

Rearranging our substitution formula to allow for some suitable differentiation operation which might lead to an expression for dx , we start off as follows:

$$e^x = u^2 - 1$$

We can get a step or two further by noting that:

$$\ln(e^x) = x$$

due to the inverse relationship between the logarithmic and exponential functions. If we therefore take natural logarithms of both sides of the above expression for e^x , we will obtain the following expression for x :

$$x = \ln|u^2 - 1|$$

which we can differentiate using the Chain Rule as follows:

$$\frac{dx}{du} = \frac{2u}{u^2 - 1} \text{ so that } dx = \frac{2u}{u^2 - 1} du$$

Substituting these conclusion back into our original integral gives us this integral in terms of u :

$$\int u \cdot \frac{2u}{u^2 - 1} du = \int \frac{2u^2}{u^2 - 1} du$$

We can make our job a little simpler by expressing our integral in terms of u as follows, using an intermediate integral I_1 :

$$I_u = 2I_1 \text{ where } I_1 = \int \frac{u^2}{u^2 - 1} du$$

We can make the integrand in I_1 a little easier to handle using the method of converting rational expressions which is explained in [Appendix 3 Section 2](#), and has been used several times throughout this book. The working is as follows:

$$\frac{u^2}{u^2 - 1} = \frac{u^2 - 1 + 1}{u^2 - 1} = \frac{u^2 - 1}{u^2 - 1} + \frac{1}{u^2 - 1} = 1 + \frac{1}{u^2 - 1}$$

This lets us rewrite I_1 in this way:

$$\int \left(1 + \frac{1}{u^2 - 1}\right) du$$

and we can deal with this term by term. The second term might be expanded using partial fractions before we finally come to do the integration, but we can in this instance call to our aid the following standard integral which was derived using partial fractions:

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

Using this standard integral (in our present case the constant $a = 1$), we can write down the following solution to our intermediate integral:

$$I_1 = u + \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| + C$$

Recalling the formula:

$$I_u = 2I_1$$

we can see that our solution in terms of u is therefore:

$$I_u = 2u + \ln \left| \frac{u-1}{u+1} \right| + C$$

and our final step will be to reverse the substitution, using the following version of our substitution formula:

$$u = \sqrt{e^x + 1}$$

We then obtain the following final solution in terms of x :

$$I = 2\sqrt{e^x + 1} + \ln \left| \frac{\sqrt{e^x + 1} - 1}{\sqrt{e^x + 1} + 1} \right| + C$$

Example 3

We will next look at another integral which bears some similarities to the integral of [Example 1](#) above, but which yields an interesting and quite different result. The integral is as follows:

$$I = \int \frac{\sqrt{x+9}}{x} dx$$

As always when we have in our integrand the square root of a linear expression (or something which looks a little like a linear expression- see the previous example), we begin with a power substitution. In this case, it will be:

$$u^2 = x + 9 \text{ so that } \sqrt{x+9} = \sqrt{u^2} = u$$

A simple rearrangement gives us:

$$x = u^2 - 9 \text{ so that } \frac{dx}{du} = 2u \text{ and so } dx = 2u du$$

We can now put all this into the original integral, to obtain the following integral in terms of u :

$$\int \frac{u}{u^2 - 9} \cdot 2u du = \int \frac{2u^2}{u^2 - 9} du = 2 \int \frac{u^2}{u^2 - 9} du$$

The integrand on the final right-hand side can be rewritten using the method of conversion of rational functions discussed in [Appendix 3 Section 2](#) in the following way:

$$\frac{u^2}{u^2 - 9} = \frac{u^2 - 9 + 9}{u^2 - 9} = \frac{u^2 - 9}{u^2 - 9} + \frac{9}{u^2 - 9} = 1 + \frac{9}{u^2 - 9}$$

This allows us to re-express our integral in terms of u in the following way:

$$2 \int \left(1 + \frac{9}{u^2 - 9} \right) du = 2 \int 1 du + 18 \int \frac{1}{u^2 - 9} du$$

The two integrals on the right-hand side can be dealt with one by one. In the case of the first one, we have:

$$\int 1 du = u + C$$

The second integral could be evaluated by partial fractions if we wished to do all the working, by first restating it with a factorised denominator, as follows:

$$\int \frac{1}{(u - 3)(u + 3)} du$$

However, we can save some effort by applying the following standard integral derived with the use of partial fractions and listed in [Appendix 4 Section 2](#):

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + C$$

In the present case, $a = 3$, which leads us to the following solution for the second of our two component integrals:

$$\int \frac{1}{u^2 - 9} du = \frac{1}{6} \ln \left| \frac{u - 3}{u + 3} \right| + C$$

Making sure we apply both the coefficients which were moved outside the integral signs in the last version of our integral in terms of u , we can now write down its solution:

$$2u + 3 \ln \left| \frac{u - 3}{u + 3} \right| + C$$

All that remains is to reverse the substitution, which was effectively as follows:

$$u = \sqrt{x + 9}$$

and we can write down this final solution in terms of x :

$$I = 2\sqrt{x+9} + 3 \ln \left| \frac{\sqrt{x+9}-3}{\sqrt{x+9}+3} \right| + C$$

One interesting aspect of this solution is that it so closely resembles the solution we obtained for [Example 2](#) above. Indeed, the solutions in terms of u are really very similar. The difference between the two integrals, however, reflects the different behaviours of the expression $e^x + 1$ in [Example 2](#) and the linear expression $x + 9$ in this example,- these differences show themselves both in terms of the algebra we perform on the substitution formula and in the way the rearranged expressions respond to the differentiation step needed to obtain expressions for dx .

Note

The above three examples all required us to carry out a little algebra - sometimes on the substitution formula, sometimes on the integral in terms of u - in order to perform the steps which would lead us to a solution. I will now present a few examples of a different kind: these examples will illustrate some situations where it would be a mistake to plunge straight into power-substitution, since a simpler route to a solution is available.

Example 4

In this example, we will evaluate the following integral:

$$I = \int \sqrt{\frac{\sin^{-1} x}{1-x^2}} dx$$

We will immediately notice that if we were to try making the substitution which straightforward power-substitution would indicate, namely the following substitution:

$$u^2 = \frac{\sin^{-1} x}{1-x^2}$$

we find ourselves faced with some real difficulties, both to rearrange the substitution formula to make x the subject, and to find an expression for dx .

Let us not be downhearted, however, since an easier approach is close at hand. Let's start by restating the integral as follows:

$$\int \frac{\sqrt{\sin^{-1} x}}{\sqrt{1-x^2}} dx \text{ or even } \int \sqrt{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}} dx$$

Once we recall the standard derivative:

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

we can readily see that the integrand contains a simple function-derivative pair. Indeed, it has the familiar form $kg'(f(x))f'(x)$, where:

$$k = 1, f(x) = \sin^{-1} x, f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$g'(f(x)) = \sqrt{\sin^{-1} x} \text{ so that } g'(x) = \sqrt{x} = x^{1/2}$$

We can easily find $g(x)$ by integrating $g'(x)$, so that:

$$g(x) = \frac{2}{3}x^{3/2} = \frac{2}{3}\sqrt{x^3}$$

The rule of integration based on the method of substitution described in [Appendix 4 Section 2](#) tells us that the solution to our integral is $kg(f(x)) + C$. In the present case, this template gives us the following solution:

$$I = \frac{2}{3}\sqrt{(\sin^{-1} x)^3} + C$$

Example 5

Let's look at another similar example. This time, our integral will be:

$$I = \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

On this occasion, there might be no obvious risk in making the expected power-substitution:

$$u^2 = x$$

However, let's pause a moment to reflect on the following standard derivative:

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

We can then make the true nature of our integrand clearer by rewriting the integral as follows:

$$\int \exp(\sqrt{x}) \cdot \frac{1}{\sqrt{x}} dx = \int 2 \exp(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} dx$$

This gives us, once again, a function-derivative pair, of the form $kg'(f(x))f'(x)$, where:

$$k = 2, f(x) = \sqrt{x}, f'(x) = \frac{1}{2\sqrt{x}}$$

$$g'(f(x)) = e^{\sqrt{x}} \text{ so that } g'(x) = e^x$$

The function $g(x)$ is easily found from $g'(x)$ in this case, and is simply:

$$g(x) = e^x$$

As before, our solution takes the form $kg(f(x)) + C$, which gives us:

$$I = 2e^{\sqrt{x}} + C$$

Example 6

A last example of this type will be given by the following integral:

$$I = \int \frac{\sqrt[3]{1 + \ln x}}{x} dx$$

It would be just about possible to work with the expected power-substitution:

$$u^3 = 1 + \ln x$$

although some challenging algebra would undoubtedly be expected.

Instead, let's notice that:

$$\frac{d}{dx}(1 + \ln x) = \frac{1}{x}$$

and then rewrite our integral as follows:

$$\int \sqrt[3]{1 + \ln x} \cdot \frac{1}{x} dx$$

This is now of the form $kg'(f(x))f'(x)$, where:

$$k = 1, f(x) = 1 + \ln x, f'(x) = \frac{1}{x} \\ g'(f(x)) = \sqrt[3]{1 + \ln x} \text{ so that } g'(x) = \sqrt[3]{x} = x^{1/3}$$

We obtain $g(x)$ from $g'(x)$ by a simple integration step, which gives us:

$$g(x) = \frac{3}{4}x^{4/3} = \frac{3}{4}\sqrt[3]{x^4}$$

and the final solution takes the form $kg(f(x)) + C$, which in this case is:

$$I = \frac{3}{4}\sqrt[3]{(1 + \ln x)^4} + C$$

Note

What we should learn from the last three examples is that when integrating irrational functions, we should not fail to look first of all for function-derivative pairs, which will allow us to use the integration rule based on integration by substitution, or, if we wish, to carry out a fairly problem-free integration by hand using substitution.

For the final example in this section, I will consider a really rather unpleasant integral, whose solution actually turns out not to be so hard after all, once we have made the right choice of substitution.

Example 7

We will take a careful look at the following rather unappetising integral:

$$I = \int \sqrt{\frac{\ln|x + \sqrt{x^2 + 1}|}{x^2 + 1}} dx$$

At first sight (and perhaps for some time thereafter) this looks rather a daunting business. If we try to make our “standard” power-substitution, we will be faced with the substitution formula:

$$u^2 = \frac{\ln|x + \sqrt{x^2 + 1}|}{x^2 + 1}$$

This presents some serious difficulties: if we want to establish the corresponding expression for dx , we must either try differentiating u in terms of x (which would require us to differentiate the integrand as it stands), or we need to make x the subject of the substitution formula before differentiating with respect to u . Neither of these would be a simple task (indeed, the second of the two options might not even be possible at all using simple algebra), so we need to think of another approach.

Falling back on the possibilities offered by substitution, then, and realising that we do not see an easily recognisable function-derivative pair which might let us use the integration rule based on the method of substitution, we have a number of other substitution options open to us. Before listing these, it will be helpful to realise that the integrand can be rewritten with two separate square-root signs, as follows:

$$\frac{\sqrt{\ln|x + \sqrt{x^2 + 1}|}}{\sqrt{x^2 + 1}}$$

Our substitution options are therefore:

$$\begin{aligned} u &= \sqrt{\ln|x + \sqrt{x^2 + 1}|} \\ u &= \ln|x + \sqrt{x^2 + 1}| \\ u &= x + \sqrt{x^2 + 1} \\ u &= \sqrt{x^2 + 1} \end{aligned}$$

$$u = x^2 + 1$$

There seems no useful trigonometric substitution indicated by the shape of the integrand: even if we were to choose something which would simplify the expression $\sqrt{x^2 + 1}$, we would still be little better off for the following reasons: the presence of an x term in the numerator would generate a sum of two trigonometric functions as the argument of a logarithmic function, and the expression for dx which would result would leave us with an unwieldy product of functions, one of which would be the multi-layered logarithmic function.

The last three substitutions in the list all present algebraic disadvantages of one sort or another; in any case, they all leave the square root of the logarithmic function fundamentally unsimplified, and would seem to require further substitutions in order to get close to a solution.

Of the top two possibilities, I will elect to try the second, since it presents a less complicated differentiation in order to get our expression for dx . We will therefore start with the following substitution:

$$u = \ln |x + \sqrt{x^2 + 1}|$$

Finding du/dx will require a judicious application of the Chain Rule: the expression for u has the form:

$$h(g(x)) \text{ where } g(x) = g_1(x) + g_2(f(x))$$

The various functions appearing in this description are as follows:

$$\begin{aligned} h(x) &= \ln|x| \\ g(x) &= g_1(x) + g_2(f(x)) \text{ where } g_1(x) = x, g_2(x) = \sqrt{x} \text{ and } g_2(f(x)) = \sqrt{x^2 + 1} \\ f(x) &= x^2 + 1 \end{aligned}$$

By the Chain Rule, the derivative of $h(g(x))$ is $h'(g(x)) \cdot g'(x)$. The function $g(x)$ can be differentiated term by term, so that:

$$g'(x) = g'_1(x) + g'_2(f(x)) \cdot f'(x)$$

where the second term is obtained using the Chain Rule. This second term on the right-hand side evaluates to:

$$\frac{1}{2\sqrt{x^2 + 1}} \cdot 2x = \frac{x}{\sqrt{x^2 + 1}}$$

so we can therefore say that:

$$g'(x) = 1 + \frac{x}{\sqrt{x^2 + 1}} = \frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}} = \frac{x + \sqrt{x^2 + 1}}{\sqrt{x^2 + 1}}$$

I have here first restated the derivative as a rational function using a single denominator, and then reversed the order of terms in the numerator to match the order in the argument of the logarithmic function contained in the original integrand.

We can now finally use the Chain Rule expression involving the functions h and g given a little earlier to write down the derivative of u in terms of x as follows:

$$\frac{du}{dx} = h'(g(x)) \cdot g'(x) = \frac{1}{x + \sqrt{x^2 + 1}} \cdot \frac{x + \sqrt{x^2 + 1}}{\sqrt{x^2 + 1}} = \frac{1}{\sqrt{x^2 + 1}}$$

We really ought to pause for a moment and contemplate the result we have just obtained; it is as follows:

$$\text{the derivative of } \ln |x + \sqrt{x^2 + 1}| \text{ is } \frac{1}{\sqrt{x^2 + 1}}$$

Who'd have thought it? Perhaps only someone who had an extraordinary talent for instant differentiation. I will return to the full implications of this in a few paragraphs. For now, however, let's write down our integral in terms of u , using the substitution we have chosen. First of all, however, we note now that:

$$\frac{du}{dx} = \frac{1}{\sqrt{x^2 + 1}} \text{ so that } dx = \sqrt{x^2 + 1} du$$

Our integral in terms of u is therefore:

$$\int \frac{\sqrt{u}}{\sqrt{x^2 + 1}} \cdot \sqrt{x^2 + 1} du = \int \sqrt{u} du = \int u^{1/2} du$$

This is integrated with remarkable ease, considering what we started with. Its solution is:

$$\frac{2}{3} u^{3/2} + C = \frac{2}{3} \sqrt{u^3} + C$$

And the final step of reversing the substitution gives this solution in terms of x :

$$I = \frac{2}{3} \sqrt{\ln^3 |x + \sqrt{x^2 + 1}|} + C \text{ or } \frac{2}{3} \sqrt{(\ln |x + \sqrt{x^2 + 1}|)^3} + C$$

(I have given two alternative versions of the notation in an attempt to satisfy all tastes!)

Some Thoughts on Example 7

I laid emphasis above, during the working for the preceding example, on the observation that:

$$\frac{d}{dx} \ln |x + \sqrt{x^2 + 1}| = \frac{1}{\sqrt{x^2 + 1}}$$

which seems on the face of it to be quite a surprising result. A consequence of it, using the Fundamental Theorem of Calculus, is that:

$$\int \frac{1}{\sqrt{x^2 + 1}} dx = \ln |x + \sqrt{x^2 + 1}| + C$$

and we might regard this as a new standard integral – but see my remarks below for more on this issue.

Having noted this relationship between the functions which appear in the numerator and denominator of our integrand, we might expect that if we express that integrand with two separate square root signs and with some separation between numerator and denominator, as follows:

$$\sqrt{\ln |x + \sqrt{x^2 + 1}|} \cdot \frac{1}{\sqrt{x^2 + 1}}$$

then we ought to be able to spot that what we have here is, in fact, a classic function-derivative pair which allows us to use our integration rule for such cases. The integrand indeed has the form:

$$kg'(f(x))f'(x)$$

where the various placeholders have the following values:

$$k = 1, g'(f(x)) = \sqrt{\ln |x + \sqrt{x^2 + 1}|}, g'(x) = \sqrt{x} = x^{1/2}$$

$$f(x) = \ln |x + \sqrt{x^2 + 1}|, f'(x) = \frac{1}{\sqrt{x^2 + 1}}$$

The solution predicted by our rule is of the form:

$$kg(f(x)) + C$$

We can obtain $g(x)$ by integrating $g'(x)$. It is as follows:

$$g(x) = \frac{2}{3}x^{3/2} = \frac{2}{3}\sqrt{x^3}$$

This allows us to write down the following solution easily:

$$\frac{2}{3}\sqrt{(\ln |x + \sqrt{x^2 + 1}|)^3} + C$$

and this is, of course, the solution we obtained by working through the substitution process above.

All this, of course, is entirely dependent on our ability to spot the relationship on which the above reasoning depends. I have not assumed this knowledge in my working since this

relationship is often unknown to students learning about integration techniques – to begin with, at least.

The relationship we have uncovered between the numerator and denominator of the integral in this example is in fact a special case of the following standard integral:

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln |x + \sqrt{x^2 + a^2}| + C$$

and I set out a derivation of this symbolic integral for reference in [Appendix 5 Section 20](#).

8 Using Multiple Techniques

In this section, we will work through a number of integration problems which call for the use of more than one of the techniques I have set out so far. We will start off with some relatively easy examples, but the level of complexity will increase quite sharply as we progress.

The first few examples are relatively straightforward, and broadly similar in nature – they will serve to present some situations where the use of substitution (other than power substitution) produces an integral in terms of u which needs a partial fractions expansion in order to allow us to integrate it. These examples differ from any we have so far looked at in that the integrands are all rational expressions containing exponential or trigonometric functions.

Example 1

Our first example will require us to integrate the following:

$$I = \int \frac{e^x}{(e^x - 1)(e^x + 3)} dx$$

We would certainly benefit from the use of a substitution which will at least remove the exponential functions from our immediate consideration, and will therefore try substituting as follows:

$$u = e^x \text{ so that } \frac{du}{dx} = e^x \text{ and } dx = \frac{1}{e^x} du = \frac{1}{u} du$$

The first form of the expression for dx should cheer us up, since it contains the promise of being able to cancel out the exponential in the numerator of our integrand. Accordingly, we apply the above information to obtain the following integral in terms of u :

$$\int \frac{u}{(u - 1)(u + 3)} \cdot \frac{1}{u} du = \int \frac{1}{(u - 1)(u + 3)} du$$

The final version of the integral in terms of u is a very ordinary example of an integral which can be evaluated using partial fractions. Indeed, it is such a simple example that we can probably feel quite justified in using the formula for integrals of the this particular form. The standard integral (listed in [Appendix 4 Section 2](#)) is as follows:

$$\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{a-b} \ln \left| \frac{x+b}{x+a} \right| + C$$

Noting simply that our variable of integration is at present u , and that in our present case the constants are as follows:

$$a = -1 \text{ and } b = 3$$

the standard integral formula gives us this result in terms of u :

$$-\frac{1}{4} \ln \left| \frac{u+3}{u-1} \right| + C$$

The opening minus sign arises because the coefficient in the integral's solution has the following form:

$$\frac{1}{a-b} = \frac{1}{-1-3} = \frac{1}{-4} = -\frac{1}{4}$$

We could have avoided this bothersome negativity by choosing our constants in reverse order, but this has only become obvious by doing the working. The current result also gives me the opportunity to point out a useful rearrangement, based on the following rule of logarithms:

$$\ln \left(\frac{a}{b} \right) = -\ln \left(\frac{b}{a} \right)$$

Our solution in terms of u can therefore be restated without a negative sign by simply exchanging the numerator and denominator of the argument of the logarithmic function, to obtain the following (much more attractive) version:

$$\frac{1}{4} \ln \left| \frac{u-1}{u+3} \right| + C$$

Reversing the substitution is particularly easy, and gives us the following final solution in x :

$$I = \frac{1}{4} \ln \left| \frac{e^x - 1}{e^x + 3} \right| + C$$

Since there was nothing especially challenging in this example, it might be worth pausing for a moment to reflect that it was made as easy as it was by the presence of an e^x term in the numerator, which allowed for cancellation with the one which made its appearance in the expression for dx . Without this term, we would have ended up with a denominator in terms of u of the following form:

$$u(u-1)(u+3)$$

The integration using partial fractions would still have been practicable (subject to the good behaviour of whatever numerator we were given instead of e^x) but the simple use of a standard integral would no longer have been open to us.

Example 2

We will now look at an example similar in appearance to the preceding one, but where no numerator term allowing for cancellation is present. Our integral will be:

$$I = \int \frac{1}{e^x + 1} dx$$

As in the last example, the most useful thing we can probably do to start with is so substitute away that exponential function. As before, this gives us:

$$u = e^x \text{ so that } \frac{du}{dx} = e^x \text{ and } dx = \frac{1}{e^x} du = \frac{1}{u} du$$

Our integral in terms of u is therefore:

$$\int \frac{1}{u+1} \cdot \frac{1}{u} du = \int \frac{1}{u(u+1)} du$$

Since the denominator of our original integrand was simpler than that of [Example 1](#), we can still apply the same standard integral we used there to the resulting integral in terms of u . In this instance, our constants will be:

$$a = 0 \text{ and } b = 1$$

giving us this first result in terms of u :

$$-\ln \left| \frac{u+1}{u} \right| + C$$

Once again, the opening minus sign arises from the following evaluation of the coefficient in the standard integral formula:

$$\frac{1}{a-b} = \frac{1}{0-1} = \frac{1}{-1} = -1$$

The rule of logarithms which came to our rescue in [Example 1](#) once again lets us rearrange our result in terms of u into the following form:

$$\ln \left| \frac{u}{u+1} \right| + C$$

and the simple reversal of the substitution gives us the following initial result in terms of x :

$$I = \ln \left(\frac{e^x}{e^x + 1} \right) + C$$

Before leaving it at that, however, let's note two more rules of logarithms – first:

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$

and second, the relationship between natural logarithms and the exponential function:

$$\ln(e^x) = x$$

Combining these two, our solution in terms of x can be restated as follows:

$$I = \ln\left(\frac{e^x}{e^x + 1}\right) + C = \ln(e^x) - \ln(e^x + 1) + C = x - \ln(e^x + 1) + C$$

Example 3

In this example, we will bid farewell to the exponential function and look at an integral which contains some trigonometric functions. This will be:

$$I = \int \frac{\sec^2 x}{\tan^3 x - \tan^2 x} dx$$

We first observe that the integrand's denominator contains two terms in $\tan x$, and the numerator comprises only $\sec^2 x$, which is the derivative of $\tan x$. We will therefore start by substituting as follows:

$$u = \tan x \text{ so that } \frac{du}{dx} = \sec^2 x \text{ and } dx = \frac{1}{\sec^2 x} du$$

Inserting these facts into the integral, we will obtain the following version in terms of u :

$$\int \frac{\sec^2 x}{u^3 - u^2} \cdot \frac{1}{\sec^2 x} du = \int \frac{1}{u^3 - u^2} du = \int \frac{1}{u^2(u - 1)} du$$

The final integrand in terms of u is amenable to an approach based on partial fractions. While it is not in the simplest form imaginable for this purpose, its denominator comprises a quadratic factor in the shape of u^2 and a linear factor $u - 1$. As the considerations summarised in [Section 3](#) above – and a quick reference to the table of partial fractions expansions in [Appendix 1 Section 3](#) – will show, we would expect to be able to expand the integrand in u in the following way:

$$\frac{1}{u^2(u - 1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u - 1}$$

In order to find the values of the placeholders A , B and C , we will create an equation of numerators. This will be as follows:

$$Au(u - 1) + B(u - 1) + Cu^2 = 1$$

Multiplying out, this turns into:

$$Au^2 - Au + Bu - B + Cu^2 = 1$$

and we can collect together like terms in u to obtain this version:

$$(A + C)u^2 + (B - A)u - B = 1$$

Equating coefficients produces the following three equations, which are easily solved:

$$\begin{aligned} A + C &= 0 \\ B - A &= 0 \\ -B &= 1 \end{aligned}$$

The third of these immediately gives us:

$$B = -1$$

Substituting this result into the second equation produces:

$$-1 - A = 0 \text{ so that } A = -1$$

and we can now substitute this into the first equation, giving:

$$-1 + C = 0 \text{ so that } C = 1$$

Our three placeholders are thus found to be:

$$A = -1, B = -1 \text{ and } C = 1$$

Putting these into the template form of our expansion which we set out above, and rearranging to remove opening minus signs, we have the following restatement of our integrand in terms of u :

$$\frac{1}{u^2(u-1)} = \frac{1}{u-1} - \frac{1}{u} - \frac{1}{u^2}$$

The integrand in terms of u can now be rewritten as follows:

$$\int \left(\frac{1}{u-1} - \frac{1}{u} - u^{-2} \right) du$$

using index notation for the last term so as to make it easier to see how we will deal with its integration.

The solution in terms of u can now be written down easily, integrating term by term – it is:

$$\ln|u-1| - \ln|u| + \frac{1}{u} + C$$

The final term in u immediately before the constant of integration is obtained as follows:

$$\int u^{-2} du = \frac{1}{-1} u^{-1} + C = -\frac{1}{u} + C$$

noting that the minus sign in front of the result will serve to transform the pre-existing minus sign in the integrand into a plus sign.

We can simplify this using the familiar rule of logarithms:

$$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$$

to reach the following neater form:

$$\ln\left|\frac{u-1}{u}\right| + \frac{1}{u} + C$$

Let's now reverse the substitution and see where that will lead. To start with, we have:

$$\ln\left|\frac{\tan x - 1}{\tan x}\right| + \frac{1}{\tan x} + C$$

It will be convenient here to state the definition of the cotangent function, which is as follows:

$$\cot x = \frac{1}{\tan x}$$

so that we can use it in the final version of our solution in terms of x . Bearing this definition in mind, we can now multiply out the numerator of the expression inside the modulus brackets to obtain the following restatement of that whole expression:

$$\frac{\tan x - 1}{\tan x} = \frac{\tan x}{\tan x} - \frac{1}{\tan x} = 1 - \cot x$$

Our solution in terms of x can now be set out as follows:

$$I = \ln|1 - \cot x| + \cot x + C$$

Some Remarks on the Above Examples

The alert reader will have perhaps noticed that the integrands in [Examples 1 and 3](#) above both contain function-derivative pairs – that is to say, they are of the form:

$$kg'(f(x))f'(x)$$

which is the standard form for basic u -substitution, and which allows us to write down the solution using the rule of integration derived from the basic method of substitution listed in

[Appendix 4 Section 2](#). We can set out the definitions of the various functions and the constant in the template I have quoted above for each of these two examples as follows:

Example 1:

$$g'(f(x)) = \frac{1}{(e^x - 1)(e^x + 3)}; g'(x) = \frac{1}{(x - 1)(x + 3)}$$

$$f(x) = e^x; f'(x) = e^x \text{ and } k = 1$$

Example 3

$$g'(f(x)) = \frac{1}{\tan^3 x - \tan^2 x}; g'(x) = \frac{1}{x^3 - x^2} = \frac{1}{x^2(x - 1)}$$

$$f(x) = \tan x; f'(x) = \sec^2 x \text{ and } k = 1$$

Using the rule of integration set out in [Appendix 4 Section 2](#), we would expect the solution of each of these integration problems to have the form:

$$kg(f(x)) + C$$

In order to create an expression for the solution, we do of course have to obtain $g(x)$ given as our starting point an expression for $g'(x)$. In many of our previous applications of this rule, this has been a fairly easy task. It requires us, in fact, to obtain the “antiderivative” of $g'(x)$, which is normally done by integration.

Applying the rule of integration to [Examples 1 and 3](#) would therefore require us to integrate the following expressions:

$$\frac{1}{(x - 1)(x + 3)} \text{ and } \frac{1}{x^2(x - 1)}$$

Both of these rational expressions require the use of partial fractions for their integration. We therefore see that, despite the apparent easy applicability of the rule of integration derived from standard u -substitution, we would still have to apply a second technique – the Method of Partial Fractions – to complete the integration.

This, of course, should make clear to us that the usefulness of the rule of integration we have been discussing here in arriving at a quick solution is dependent on the ease with which the function $g'(x)$ in the standard model of the integrand can itself be integrated. If this can be done “at sight”, then the rule can be really effective. If further work is required to complete the integration, then the rule only takes us part of the way to our final result.

Because the rule of integration alone was insufficient for the job, I chose in the above examples to carry out the substitutions “by hand”, to show how these examples qualify as requiring more than one technique.

We will now proceed to examine some more complicated examples of the use of multiple integration techniques.

Example 4

In this example, we will deal with an integral which looks relatively inoffensive at first, but which soon proves a little more interesting than we might have thought. The integral is as follows:

$$I = \int \sin \sqrt{x-1} \, dx$$

There is a square root present here, so that we might first think that power substitution is the right approach. Indeed, it will help us, and is probably an essential starting point. For the first time in this book, I will use a substitution variable of t rather than the more customary u . My reasons for doing so will, I hope, become obvious very shortly.

The usual power substitution to take the square root out of consideration would be:

$$t^2 = x - 1 \text{ so that } x = t^2 + 1$$

We also remark, in our usual justification of a power substitution, that this choice means that:

$$\sqrt{x-1} = \sqrt{t^2} = t$$

Furthermore, differentiation of the above expression for x in terms of t gives the following:

$$\frac{dx}{dt} = 2t \text{ so that } dx = 2t \, dt$$

All of these items of information can now be applied to our original integral to obtain the following integral in terms of t :

$$\int \sin t \cdot 2t \, dt = 2 \int t \sin t \, dt$$

Leaving out of consideration for a short while the coefficient of 2 which we have moved outside the integral sign, we are left with an integral in terms of t which is of exactly the kind we looked at extensively in the early sections of [Part 2](#) – where we used integration by parts to integrate expressions comprising a product of simpler expressions of two different kinds. Accordingly, we will move on to evaluate the integral in terms of t by choosing parts with the aid of our old friend the **LIATE** rule, as follows:

$$u = t \text{ and } \frac{dv}{dt} = \sin t$$

(You will now, I hope, see why I chose a substitution variable of t in the earlier working – in order to be able to use the “usual” names u and v for integration by parts without confusing the first of these with the earlier substitution variable.) These choices of parts have the following consequences:

$$\frac{du}{dt} = 1 \text{ and } v = -\cos t$$

The integration by parts formula now gives us the following result:

$$\int t \sin t \, dt = -t \cos t + \int \cos t \, dt = \sin t - t \cos t + C$$

To arrive at the final right-hand side, we integrate $\cos t$ and then reverse the order of the terms to avoid an opening minus sign. Making sure we restore the coefficient of 2 which lay outside the integral sign a little earlier, our overall solution in terms of the substitution variable t is as follows:

$$2 \sin t - 2t \cos t + C$$

and reversing the substitution using the equation:

$$t = \sqrt{x-1}$$

which we obtained above from our original substitution formula gives us the following final solution in terms of x :

$$I = 2 \sin \sqrt{x-1} - 2\sqrt{x-1} \cos \sqrt{x-1} + C$$

Power substitution was clearly the correct choice as our opening move. It produced an integral in terms of t which happened to require integration by parts for its solution. It was, of course, a relatively simple example of an integral requiring this technique, and so caused us little difficulty. This is, however, the first integral we have presented in this book which was evaluated using a combination of substitution and integration by parts.

Example 5

We will now proceed to an example which has some similarities to the previous one, at least in terms of the approach which will be needed to evaluate it. Our integral will be the following:

$$I = \int \frac{x^2}{\sqrt{x^2-4}} dx$$

The first thing to notice here is the presence in the integrand of a denominator of the form:

$$\sqrt{x^2 - a^2}$$

where in our case the constant $a = 2$. We saw in [Part 1 Section 11](#) that a useful approach where this kind of sub-expression is present is secant-substitution. Here, therefore, we will start by trying the substitution:

$$x = 2 \sec u \text{ so that } \frac{dx}{du} = 2 \sec u \tan u \text{ and } dx = 2 \sec u \tan u du$$

We note furthermore how this substitution transforms the expression under the square root sign in our denominator (with the additional help of a trigonometric identity):

$$x^2 - 4 = 4\sec^2 u - 4 = 4(\sec^2 u - 1) = 4\tan^2 u$$

Consequently, the denominator as a whole becomes:

$$\sqrt{4\tan^2 u} = 2 \tan u$$

We can now put all this back into the original integral to obtain the following integral in terms of u :

$$\int \frac{4\sec^2 u}{2 \tan u} \cdot 2 \sec u \tan u du = \int \frac{4\sec^3 u \tan u}{\tan u} du = \int 4\sec^3 u du = 4 \int \sec^3 u du$$

We now see that we are left with the problem of integrating $\sec^3 u$. We have already presented the integration of this function in [Part 2 Section 12](#) of this book: we divided the integrand into the product of two functions and applied integration by parts. Together with the use of a trigonometric identity, this provided us with an equation for the solution, which we solved to give the following solution (changing the variable of integration from x to the present variable u):

$$\int \sec^3 u du = \frac{1}{2} \sec u \tan u + \frac{1}{2} \ln|\sec u + \tan u| + C$$

If we now reapply the coefficient of 4 which we moved outside the integral sign, then with the benefit of this result from [Part 2](#), the solution to our current integral in terms of u becomes:

$$2 \sec u \tan u + 2 \ln|\sec u + \tan u| + C$$

Preparing to reverse the substitution, we recall that our original substitution formula was expressed in the “reverse” form – we chose:

$$x = 2 \sec u \text{ so that } u = \sec^{-1} \frac{1}{2} x$$

The reversal of the substitution then gives us the following rather unwieldy expression for our solution in terms of x :

$$2 \sec \sec^{-1} \frac{1}{2} x \tan \sec^{-1} \frac{1}{2} x + 2 \ln \left| \sec \sec^{-1} \frac{1}{2} x + \tan \sec^{-1} \frac{1}{2} x \right| + C$$

We can take a first step towards simplifying this by noting that:

$$\sec \sec^{-1} x = x$$

So that the solution can be written as:

$$x \tan \sec^{-1} \frac{1}{2}x + 2 \ln \left| \frac{1}{2}x + \tan \sec^{-1} \frac{1}{2}x \right| + C$$

To complete the job, we need to recall the result we quoted in [Part 1 Section 11 Example 3](#) (illustrating secant-substitution) to the effect that:

$$\tan \sec^{-1} x = \sqrt{x^2 - 1}$$

(This is one of a number of rather curious results allowing us to simplify compounds of trigonometric and inverse trigonometric functions into a form which excludes any trigonometric function, which I have quoted in the main text and derived fully in [Appendix 4 Section 4](#).) With the aid of this result, we can now restate our solution in the following form:

$$x \sqrt{\frac{x^2}{4} - 1} + 2 \ln \left| \frac{1}{2}x + \sqrt{\frac{x^2}{4} - 1} \right| + C$$

This is all very well as it stands, but we might well be inclined to ask whether we can simplify the expression which appears under each of the square roots in the solution. Fortunately, this is not too difficult – we can rewrite this expression as one single rational expression with a denominator of 4, as follows:

$$\frac{x^2}{4} - 1 = \frac{x^2}{4} - \frac{4}{4} = \frac{x^2 - 4}{4}$$

(This can be seen as the reverse process corresponding to the operation I have called “converting rational expressions” and set out as an algebraic technique in [Appendix 3 Section 2](#).) Restoring the square root, we will observe a further useful simplification as follows:

$$\sqrt{\frac{x^2}{4} - 1} = \sqrt{\frac{x^2 - 4}{4}} = \frac{\sqrt{x^2 - 4}}{\sqrt{4}} = \frac{1}{2} \sqrt{x^2 - 4}$$

The solution in terms of x has now become a little less unattractive, and now takes the following form:

$$I = \frac{1}{2}x\sqrt{x^2 - 4} + 2 \ln \left| \frac{1}{2}x + \frac{1}{2}\sqrt{x^2 - 4} \right| + C$$

There is one more step we can take, however. If we rewrite the second, logarithmic, term in the above solution by factorising out $1/2$ from the expression inside the modulus brackets, we will obtain the following:

$$2 \ln \left| \frac{1}{2}x + \frac{1}{2}\sqrt{x^2 - 4} \right| = 2 \ln \left| \frac{1}{2}(x + \sqrt{x^2 - 4}) \right|$$

We can now apply this familiar rule of logarithms:

$$\ln(ab) = \ln a + \ln b$$

to go one step further and restate the logarithmic term in two components, as follows:

$$2 \ln \left(\frac{1}{2} \right) + 2 \ln |x + \sqrt{x^2 - 4}|$$

Now the first term in the expression above is a constant. We can therefore take the step of “restating the constant” which we have done on one or two occasions already in this book. We restate the constant of integration as follows:

$$C' = C + 2 \ln \left(\frac{1}{2} \right)$$

This allows us to write down our final solution in terms of x as follows

$$I = \frac{1}{2} x \sqrt{x^2 - 4} + 2 \ln |x + \sqrt{x^2 - 4}| + C'$$

Before leaving this example, let’s briefly review the various techniques we have applied in obtaining this solution. First, we used secant-substitution, inspired by the square root expression in the denominator of the integrand. We then simplified the integrand in terms of u using the trigonometric identity which connects $\sec x$ and $\tan x$, and obtained an integrand in terms of u which required integration by parts and the solution of the resulting equation for the result. Reversing the substitution needed one of the identities which allows the simplification of compounds of a trigonometric function and an inverse trigonometric function, further algebraic simplification of the expression under the square root signs, and finally a restatement of the constant of integration. This is quite a roster of techniques – possibly the lengthiest so far; furthermore, this constitutes another integral which can be solved using substitution followed by integration by parts – like the preceding example, but perhaps a little more involved.

Example 6

For our last example requiring multiple techniques, we will face a truly fearsome adversary – we will integrate the following cube root expression:

$$I = \int \sqrt[3]{\frac{x+1}{x-1}} dx$$

so I would advise you to fasten your safety belts.

We will begin without further ado by proposing power-substitution as the easiest way of at least getting the cube root out of our way. Our usual approach to choosing a power substitution would give us the following:

$$u^3 = \frac{x+1}{x-1} \text{ so that } \sqrt[3]{\frac{x+1}{x-1}} = \sqrt[3]{u^3} = u$$

It is not clear, however, how we might go about obtaining an expression for dx without a complicated differentiation requiring both the Quotient Rule and the Chain Rule. I recommend instead the following algebraic rearrangement of the right-hand side of the substitution formula, based on the conversion of rational expressions as set out in [Appendix 3 Section 2](#). We can write:

$$\frac{x+1}{x-1} = \frac{x-1+2}{x-1} = \frac{x-1}{x-1} + \frac{2}{x-1} = 1 + \frac{2}{x-1}$$

We can now perform the following rearrangements on the substitution formula:

$$u^3 - 1 = \frac{2}{x-1} \text{ so that } \frac{x-1}{2} = \frac{1}{u^3-1} \text{ and } x-1 = \frac{2}{u^3-1} \text{ and } x = \frac{2}{u^3-1} + 1$$

We are now able to obtain our expression for dx using a rather simpler application of the Quotient Rule than would have been the case earlier:

$$\frac{dx}{du} = \frac{0 - 2 \cdot 3u^2}{(u^3 - 1)^2} = -\frac{6u^2}{(u^3 - 1)^2} \text{ so that } dx = -\frac{6u^2}{(u^3 - 1)^2} du$$

Our integral in terms of u is therefore now as follows:

$$I_u = \int u \cdot \left(-\frac{6u^2}{(u^3 - 1)^2} \right) du = -6 \int \frac{u^3}{(u^3 - 1)^2} du$$

This gives us an integrand which is perhaps only mildly less disturbing than the one we began with. We will address this question in a moment. First of all, however, we will define the integral in u above as an intermediate integral I_1 in the following way:

$$I_1 = \int \frac{u^3}{(u^3 - 1)^2} du$$

This means that if we name the integral in terms of u as I_u , we can then record the following formula for our eventual solution:

$$I_u = -6I_1$$

This simple convention will make the remainder of our task that little bit easier (and we may need all the help we can get).

We note that we can now rearrange the integrand of I_1 as follows:

$$\frac{u^3}{(u^3 - 1)^2} = \frac{u^3 - 1 + 1}{(u^3 - 1)^2} = \frac{u^3 - 1}{(u^3 - 1)^2} + \frac{1}{(u^3 - 1)^2} = \frac{1}{u^3 - 1} + \frac{1}{(u^3 - 1)^2}$$

This is a similar process to the other conversions of rational expressions we have displayed elsewhere in this book.

Neither of the terms above look all that easy to integrate in themselves, however, so we have to go further. Fortunately, we have at our disposal the tremendously useful rule of algebra for the difference of two cubes:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

This lets us rewrite the integrand of I_1 in the following slightly more convenient form:

$$\frac{1}{(u-1)(u^2+u+1)} + \frac{1}{(u-1)^2(u^2+u+1)^2}$$

Here we have two terms, both of which have denominators which place them within the range of our partial fractions methods: the first denominator has one linear term and one quadratic term, while the second has a repeated linear term and a repeated quadratic term. A quick look at [Appendix 1 Section 3](#) will confirm that we have the necessary techniques at our disposal to produce partial fractions expansions for both terms which will help us to integrate them.

It's worth a quick remark to the effect that we cannot factorise the denominators in either of our two terms any further: the quadratic factors in each are one and the same, and they have the usual form of a quadratic expression, that is:

$$ax^2 + bx + c \text{ where } a = 1, b = 1 \text{ and } c = 1$$

As a result, the discriminant of the quadratic is:

$$b^2 - 4ac = 1^2 - 4 \cdot 1 \cdot 1 = 1 - 4 = -3$$

The value is less than zero, so that the quadratic is shown to have no real roots, and we cannot factorise it any further using real numbers. It is therefore an *irreducible* quadratic.

We have been able to split the integrand of I_1 into two terms, so that it can now be expressed as follows:

$$I_1 = \int \frac{1}{(u-1)(u^2+u+1)} du + \int \frac{1}{(u-1)^2(u^2+u+1)^2} du$$

If we make things a little more convenient by naming these two integrals as follows:

$$I_2 = \int \frac{1}{(u-1)(u^2+u+1)} du \text{ and } I_3 = \int \frac{1}{(u-1)^2(u^2+u+1)^2} du$$

then we can express our earlier intermediate integrals in terms of these two, giving us:

$$I_1 = I_2 + I_3 \text{ so that } I_u = -6(I_2 + I_3)$$

We will now proceed to obtain partial fractions expansions for each of these two integrands; this will allow us to proceed further towards their integration.

Example 6: Expansion of the Integrand of I_2

The integrand of our intermediate integral I_2 is as follows:

$$\frac{1}{(u-1)(u^2+u+1)}$$

If we look at the table given in the reference section on partial fractions in [Appendix 1 Section 3](#), we see that this has the form:

$$\frac{px^2 + qx + r}{(x+a)(x^2+bx+c)}$$

The numerator in our present case, however, is about as simple as it could get – the constants in the template given above have the following values:

$$p = 0, q = 0 \text{ and } r = 1$$

The table also shows that we would expect our partial fractions expansion to be of the form:

$$\frac{A}{u-1} + \frac{Bu+C}{u^2+u+1}$$

Expressing the above expansion as a single rational expression with a common denominator and then equating numerators gives us the following equation:

$$A(u^2+u+1) + (Bu+C)(u-1) = 1$$

Multiplying out all the bracketed items gives us this:

$$Au^2 + Au + A + Bu^2 - Bu + Cu - C = 1$$

If we gather together like terms, then we obtain:

$$(A+B)u^2 + (A-B+C)u + A-C = 1$$

and we can then equate coefficients to derive the following three equations in our three placeholder variables:

$$\begin{aligned} A+B &= 0 \\ A-B+C &= 0 \\ A-C &= 1 \end{aligned}$$

The third of these equations can be rearranged to give us this expression for C in terms of A :

$$C = A - 1$$

We can substitute this into the second equation to obtain:

$$A - B + (A - 1) = 0 \text{ or } 2A - B = 1$$

We now have two equations in just A and B , which are as follows:

$$\begin{aligned} A + B &= 0 \\ 2A - B &= 1 \end{aligned}$$

The first of these rearranges to give us:

$$B = -A$$

And we can substitute this into our second equation in the above pair to obtain:

$$2A + A = 1 \text{ or } 3A = 1 \text{ so that } A = \frac{1}{3}$$

We now recall the earlier equations we obtained with B and C as their subject. These were:

$$B = -A \text{ and } C = A - 1$$

Substituting in the value we have obtained for A we then have:

$$B = -\frac{1}{3} \text{ and } C = -\frac{2}{3}$$

We now have the following partial fractions expansion for the integrand of I_2 :

$$\frac{\frac{1}{3}}{u-1} - \frac{\frac{1}{3}u + \frac{2}{3}}{u^2 + u + 1}$$

This can be rendered much less of a mess by factorising out the common factor of $1/3$ to give us the following version:

$$\frac{1}{3} \left(\frac{1}{u-1} - \frac{u+2}{u^2 + u + 1} \right)$$

Example 6: Expansion of the Integrand of I_3

The intermediate integral I_3 has the following integrand:

$$\frac{1}{(u-1)^2(u^2 + u + 1)^2}$$

This is similar to the integrand of I_2 , which we have just considered – in the current case, each term of the denominator of the integrand of I_2 is present in our denominator, but each of these terms is *squared*, in other words, each one is present but repeated. I remarked a little

earlier that the table of partial fractions expansions presented in [Appendix 1 Section 3](#) would enable us to expand this integrand – this statement requires a little modification, however, now that we have come to the actual task, since the table does not explicitly deal with the situation of a denominator which contains both a repeated linear factor and a repeated irreducible quadratic factor.

We can, however, infer a suitable expansion from the actual information in the table as it stands. The expansion shown in the fourth row of the table indicates that the repeated linear factor would be expected to give us the following terms in our expansion (I have chosen the second of the two alternative expansions given, since it will make the eventual integration easier):

$$\frac{A}{u-1} + \frac{B}{(u-1)^2}$$

The contribution made by the repeated quadratic factor can similarly be inferred from the expansion given in the last row of the table in [Appendix 1 Section 3](#). We can expect that contribution to be as follows:

$$\frac{Cu + D}{u^2 + u + 1} + \frac{Eu + F}{(u^2 + u + 1)^2}$$

We can put these together to obtain the following template for a partial fractions expansion for the integrand of I_3 :

$$\frac{1}{(u-1)^2(u^2 + u + 1)^2} = \frac{A}{u-1} + \frac{B}{(u-1)^2} + \frac{Cu + D}{u^2 + u + 1} + \frac{Eu + F}{(u^2 + u + 1)^2}$$

In order to obtain actual values for the placeholders A to F on the right-hand side, we must restate the right-hand side as a single rational expression with a common denominator and then establish an equation between the numerators on the left- and right-hand sides. We have done this sufficiently often in this part of the book that it should be a straightforward process. With the appropriate common denominator, the right-hand side can be restated as follows:

$$\frac{A(u-1)(u^2 + u + 1)^2 + B(u^2 + u + 1)^2 + (Cu + D)(u-1)^2(u^2 + u + 1) + (Eu + F)(u-1)^2}{(u-1)^2(u^2 + u + 1)^2}$$

We must now set about carefully multiplying out the various terms in the above numerator. Let's take them one at a time, dealing first with the squared components of each.

We first note that:

$$(u^2 + u + 1)^2 = (u^2 + u + 1)(u^2 + u + 1)$$

The right-hand side can be rewritten as:

$$u^2(u^2 + u + 1) + u(u^2 + u + 1) + (u^2 + u + 1)$$

which multiplies out as follows:

$$u^4 + u^3 + u^2 + u^3 + u^2 + u + u^2 + u + 1$$

and we can gather together like terms to arrive at the following:

$$(u^2 + u + 1)^2 = u^4 + 2u^3 + 3u^2 + 2u + 1$$

The other squared factor in the above expansion multiplies out very easily as follows:

$$(u - 1)^2 = u^2 - 2u + 1$$

We can now multiply out the coefficient of the placeholder A in the numerator above' it is:

$$(u - 1)(u^2 + u + 1)^2 = (u - 1)(u^4 + 2u^3 + 3u^2 + 2u + 1)$$

The right-hand side can be rewritten as:

$$u(u^4 + 2u^3 + 3u^2 + 2u + 1) - (u^4 + 2u^3 + 3u^2 + 2u + 1)$$

and this expands out to the following:

$$u^5 + 2u^4 + 3u^3 + 2u^2 + u - u^4 - 2u^3 - 3u^2 - 2u - 1$$

Gathering together like terms gives us the following:

$$u^5 + u^4 + u^3 - u^2 - u - 1$$

The coefficient of B is the expansion of the squared quadratic obtained a few lines ago, namely:

$$u^4 + 2u^3 + 3u^2 + 2u + 1$$

The coefficient of $Cu + D$ is a harder nut to crack, being equivalent to the following product of two terms:

$$(u^2 - 2u + 1)(u^2 + u + 1)$$

We can rewrite this, using the same approach as we have already taken, in the following form:

$$u^2(u^2 + u + 1) - 2u(u^2 + u + 1) + (u^2 + u + 1)$$

This multiplies out to give the following:

$$u^4 + u^3 + u^2 - 2u^3 - 2u^2 - 2u + u^2 + u + 1$$

and we can gather together like terms to arrive at the following result:

$$u^4 - u^3 - u + 1$$

The coefficient of $Eu + F$, the square of a linear expression, can be written out very easily, as we have already seen, as follows:

$$u^2 - 2u + 1$$

We are now in a position to summarise the multiplied out versions of the coefficients of each of our four placeholder expressions. These are as follows:

A is multiplied by $u^5 + u^4 + u^3 - u^2 - u - 1$

B is multiplied by $u^4 + 2u^3 + 3u^2 + 2u + 1$

$Cu + D$ is multiplied by $u^4 - u^3 - u + 1$

$Eu + F$ is multiplied by $u^2 - 2u + 1$

We now proceed to multiply out each of the four terms whose contents have been summarised above. The first two are easy – they give us, respectively:

$$Au^5 + Au^4 + Au^3 - Au^2 - Au - A$$

and:

$$Bu^4 + 2Bu^3 + 3Bu^2 + 2Bu + B$$

The third and fourth terms have to be multiplied out in stages; the third term is as follows:

$$(Cu + D)(u^4 - u^3 - u + 1)$$

This can be rewritten as:

$$Cu(u^4 - u^3 - u + 1) + D(u^4 - u^3 - u + 1)$$

which multiplies out as follows:

$$Cu^5 - Cu^4 - Cu^2 + Cu + Du^4 - Du^3 - Du + D$$

Gathering together like terms gives the following result:

$$Cu^5 + (D - C)u^4 - Du^3 - Cu^2 + (C - D)u + D$$

Taking the same approach to the fourth term, we begin with:

$$(Eu + F)(u^2 - 2u + 1)$$

We rewrite this expression as:

$$Eu(u^2 - 2u + 1) + F(u^2 - 2u + 1)$$

Multiplying out gives us:

$$Eu^3 - 2Eu^2 + Eu + Fu^2 - 2Fu + F$$

Gathering together like terms, we arrive at the following:

$$Eu^3 + (F - 2E)u^2 + (E - 2F)u + F$$

Our four terms have now been multiplied out and fully simplified to give the following results:

$$\begin{aligned} & Au^5 + Au^4 + Au^3 - Au^2 - Au - A \\ & Bu^4 + 2Bu^3 + 3Bu^2 + 2Bu + B \\ & Cu^5 + (D - C)u^4 - Du^3 - Cu^2 + (C - D)u + D \\ & Eu^3 + (F - 2E)u^2 + (E - 2F)u + F \end{aligned}$$

In the numerator we are constructing, all of these four terms are to be added together. We can therefore cut the process short by simply gathering together like terms from all four of the above expressions and writing the results down for each power of u in the numerator we are constructing. The results are as follows:

$$\begin{aligned} u^5: & A + C \\ u^4: & A + B - C + D \\ u^3: & A + 2B - D + E \\ u^2: & 3B - A - C - 2E + F \\ u: & 2B - A + C - D + E - 2F \\ 1: & B - A + D + F \end{aligned}$$

The numerator constructed according to the pattern I have just set out must be equated to the numerator of the original rational expression, which is 1. As a consequence, we find that the coefficient of every power of u must be equated to zero, while the constant part of the numerator we have constructed (in other words, the expression which is the coefficient of 1 in the above list) is equated to 1.

This reasoning gives us the following set of linear equations – six equations in six variables:

$$\begin{aligned} A + C &= 0 \\ A + B - C + D &= 0 \\ A + 2B - D + E &= 0 \\ 3B - A - C - 2E + F &= 0 \\ 2B - A + C - D + E - 2F &= 0 \\ B - A + D + F &= 1 \end{aligned}$$

Solving these seems like quite a tall order. However, we have the necessary methods at our disposal – we have solved systems of linear equations elsewhere in this book – in particular, the system of five equations we had to deal with in [Section 2 Example 4](#) above – so we will endeavour to keep our heads and work through the process of solution in a systematic way.

To start with, then, the first equation above gives us a simple way of expressing A in terms of one of the other placeholders; it can be rearranged to give:

$$A = -C$$

Substituting this expression for A into the other five equations gives us this revised system of equations in only five variables:

$$\begin{aligned}B - 2C + D &= 0 \\2B - C - D + E &= 0 \\3B - 2E + F &= 0 \\2B + 2C - D + E - 2F &= 0 \\B + C + D + F &= 1\end{aligned}$$

The first of these equations can be rearranged to give the following expression for B :

$$B = 2C - D$$

And we can once again substitute this expression into the remaining four equations to obtain this smaller system of equations:

$$\begin{aligned}3C - 3D + E &= 0 \\6C - 3D - 2E + F &= 0 \\6C - 3D + E - 2F &= 0 \\3C + F &= 1\end{aligned}$$

We could now go right ahead following the ascending order of our placeholder letters and obtain an expression for C in terms of the other placeholders. However, all the instances of the placeholder C in the above system of four equations has a coefficient other than 1, so that any such equation at this point would be likely to introduce fractional coefficients and make our arithmetic trickier. In view of this, I will now move to the other end of the list of our six placeholders, and rearrange the last equation in the above list to obtain the following expression for F :

$$F = 1 - 3C$$

We can substitute this into the first three equations to get the following system of three equations:

$$\begin{aligned}3C - 3D + E &= 0 \\3C - 3D - 2E &= -1 \\12C - 3D + E &= 2\end{aligned}$$

The first equation in this system can be rearranged to give:

$$E = 3D - 3C$$

and we can substitute this into the remaining two to obtain:

$$\begin{aligned}9C - 9D &= -1 \\9C &= 2\end{aligned}$$

This is now looking a great deal simpler. The second equation above gives us an immediate value for C , namely:

$$C = \frac{2}{9}$$

And the first equation can be rearranged to give an expression for D in terms of C , as follows:

$$9D = 9C + 1 \text{ so that } D = C + \frac{1}{9}$$

so we see that we are going to have to do some fractional arithmetic after all – at least we have left it all to the end.

Before evaluating all our placeholders, let's summarise the expressions for each which we derived at the various stages of our solution process above; they are as follows – in reverse order:

$$\begin{aligned} D &= C + \frac{1}{9} \\ E &= 3D - 3C \\ F &= 1 - 3C \\ B &= 2C - D \\ A &= -C \end{aligned}$$

We can now apply each of these in turn, beginning with the substitution into the first equation of the value we have just obtained for C .. In order, these give us:

$$\begin{aligned} D &= \frac{2}{9} + \frac{1}{9} = \frac{3}{9} = \frac{1}{3} \\ E &= \frac{3.1}{3} - \frac{3.2}{9} = \frac{3}{3} - \frac{6}{9} = \frac{9-6}{9} = \frac{3}{9} = \frac{1}{3} \\ F &= 1 - \frac{3.2}{9} = \frac{9-6}{9} = \frac{3}{9} = \frac{1}{3} \\ B &= \frac{2.2}{9} - \frac{1}{3} = \frac{4-3}{9} = \frac{1}{9} \\ A &= -\frac{2}{9} \end{aligned}$$

We can now put these values back into the template for our partial fractions expansion which we obtained a little earlier. We will arrive at the following expansion for the integrand of I_3 :

$$\frac{\frac{1}{9}}{(u-1)^2} - \frac{\frac{2}{9}}{u-1} + \frac{\frac{2}{9}u + \frac{1}{3}}{u^2 + u + 1} + \frac{\frac{1}{3}u + \frac{1}{3}}{(u^2 + u + 1)^2}$$

Expressing all the fractions in the numerator with the common denominator of 9, this becomes:

$$\frac{\frac{1}{9}}{(u-1)^2} - \frac{\frac{2}{9}}{u-1} + \frac{\frac{2}{9}u + \frac{3}{9}}{u^2 + u + 1} + \frac{\frac{3}{9}u + \frac{3}{9}}{(u^2 + u + 1)^2}$$

We can now factorise out a common factor of $1/9$, and display our result in the neater form:

$$\frac{1}{9} \left(\frac{1}{(u-1)^2} - \frac{2}{u-1} + \frac{2u+3}{u^2+u+1} + \frac{3u+3}{(u^2+u+1)^2} \right)$$

Example 6: A Pause for Breath

Having created two rather complicated partial fractions expansions, we ought to pause briefly to summarise our progress so far.

We have expanded the integrands of the intermediate integrals I_2 and I_3 as follows:

$$\frac{1}{(u-1)(u^2+u+1)} = \frac{1}{3} \left(\frac{1}{u-1} - \frac{u+2}{u^2+u+1} \right)$$

and:

$$\frac{1}{(u-1)^2(u^2+u+1)^2} = \frac{1}{9} \left(\frac{1}{(u-1)^2} - \frac{2}{u-1} + \frac{2u+3}{u^2+u+1} + \frac{3u+3}{(u^2+u+1)^2} \right)$$

As a result of this preparatory work, we can now divide up the intermediate integrals I_2 and I_3 in the following manner:

$$I_2 = \frac{1}{3} \left(\int \frac{1}{u-1} du - \int \frac{u+2}{u^2+u+1} du \right)$$

and:

$$I_3 = \frac{1}{9} \left(\int \frac{1}{(u-1)^2} du - \int \frac{2}{u-1} du + \int \frac{2u+3}{u^2+u+1} du + \int \frac{3u+3}{(u^2+u+1)^2} du \right)$$

We can therefore define some more intermediate integrals as follows – first, the two components of the expanded I_2 :

$$I_4 = \int \frac{1}{u-1} du \text{ and } I_5 = \int \frac{u+2}{u^2+u+1} du$$

and now the four components of the expanded version of I_3 :

$$I_6 = \int \frac{1}{(u-1)^2} du, I_7 = \int \frac{2}{u-1} du, I_8 = \int \frac{2u+3}{u^2+u+1} du, I_9 = \int \frac{3u+3}{(u^2+u+1)^2} du$$

These additional intermediate integrals are then related to our original two, I_2 and I_3 , by the following two formulae:

$$I_2 = \frac{1}{3}(I_4 - I_5) \text{ and } I_3 = \frac{1}{9}(I_6 - I_7 + I_8 + I_9)$$

It will be useful to combine these two formulae into one formula for the integral in terms of u which we obtained from our power-substitution; we will do this by recalling the earlier formula for I_u , which was:

$$I_u = -6(I_2 + I_3)$$

Substituting into this formula the expressions we have just set out for I_2 and I_3 , we obtain:

$$I_u = -6\left(\frac{1}{3}(I_4 - I_5) + \frac{1}{9}(I_6 - I_7 + I_8 + I_9)\right)$$

The right-hand side can be multiplied out so that we can write:

$$I_u = -2(I_4 - I_5) - \frac{2}{3}(I_6 - I_7 + I_8 + I_9)$$

and we can abolish the unpleasant opening minus sign by reordering the bracketed material in the first term to give:

$$I_u = 2(I_5 - I_4) - \frac{2}{3}(I_6 - I_7 + I_8 + I_9)$$

We can now proceed to integrate each of the intermediate integrals in the above composite formula, putting their solutions together at the end to obtain our solution in terms of u . We will make it easier to avoid getting lost in complexity if we go further, and identify each of the two terms in the above formula using the following abbreviations:

$$J_1 = 2(I_5 - I_4) \text{ and } J_2 = \frac{2}{3}(I_6 - I_7 + I_8 + I_9)$$

Using these abbreviations, we can then present the following shorthand formula for the integral in terms of u :

$$I_u = J_1 - J_2$$

Example 6: Integrating I_4 and I_5 to Evaluate J_1

We now need to integrate the two intermediate integrals I_4 and I_5 . The first of these is fairly easy. We recall that:

$$I_4 = \int \frac{1}{u-1} du$$

so that the rule of integration we have used again and again in this part of the book gives us this solution:

$$I_4 = \ln|u-1| + C$$

The second of these two intermediate integrals is:

$$I_5 = \int \frac{u+2}{u^2+u+1} du$$

In [Appendix 4 Section 2](#), I quote a “standard integral” which can be used to deal with rational expressions of the above kind which contain an irreducible quadratic denominator. We have seen expressions of this kind several times before, particularly when we were dealing with polynomial division as a useful tool in integration. The standard integral is derived in detail in [Appendix 5 Section 19](#), and it can be stated as follows:

$$\int \frac{px+q}{x^2+bx+c} dx = \frac{p}{2} \ln|x^2+bx+c| - \frac{bp-2q}{\sqrt{4c-b^2}} \tan^{-1} \frac{2}{\sqrt{4c-b^2}} \left(x + \frac{b}{2}\right) + C$$

In order to confirm its applicability here, we note that we have already established that the denominator is an irreducible quadratic – this is, of course, a requirement for the standard integral we want to use to be available to us. In order to apply the formula, we note that the values of the various constant coefficients on the right-hand side are in this case as follows:

$$p = 1, q = 2, b = 1 \text{ and } c = 1$$

The two rather complicated coefficients on the right-hand side can therefore be evaluated as follows:

$$\begin{aligned} \frac{bp-2q}{\sqrt{4c-b^2}} &= \frac{1.1-2.2}{\sqrt{4.1-1^2}} = -\frac{3}{\sqrt{3}} = -\sqrt{3} \\ \frac{2}{\sqrt{4c-b^2}} &= \frac{2}{\sqrt{4.1-1^2}} = \frac{2}{\sqrt{3}} \end{aligned}$$

Our solution can now be read off from the standard integral formula as follows:

$$\frac{1}{2} \ln|u^2+u+1| + \sqrt{3} \tan^{-1} \frac{2}{\sqrt{3}} \left(u + \frac{1}{2}\right) + C$$

Having reached this result, we might like to neaten up the argument of the inverse tangent by multiplying it out – this is done as follows:

$$\frac{2}{\sqrt{3}} \left(u + \frac{1}{2}\right) = \frac{2u}{\sqrt{3}} + \frac{2}{2\sqrt{3}} = \frac{2u}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{2u+1}{\sqrt{3}}$$

and it gives us the following version of the solution:

$$I_5 = \frac{1}{2} \ln|u^2+u+1| + \sqrt{3} \tan^{-1} \left(\frac{2u+1}{\sqrt{3}}\right) + C$$

From these two results, the first term in the expanded formula for I_u which we derived a little earlier and is formed as follows:

$$J_1 = 2(I_5 - I_4)$$

evaluates to the following expression:

$$J_1 = \ln|u^2 + u + 1| + 2\sqrt{3} \tan^{-1}\left(\frac{2u + 1}{\sqrt{3}}\right) - 2 \ln|u - 1|$$

Perhaps I may be forgiven for leaving the constant of integration out of consideration for now, since this forms only one component of our substantive solution.

Example 6: Integrating I_6 to I_8 - Evaluating J_1 , Part 1

We can now move on to deal successively with each of the intermediate integrals which appear in the second term of our formula for I_u , beginning with this one:

$$I_6 = \int \frac{1}{(u - 1)^2} du$$

This integral can be rewritten using a negative power, as follows:

$$\int (u - 1)^{-2} du$$

Once this is done, we see that one of the rules of integration we derived from the technique of substitution can be applied – here is the rule in question:

$$\int (x + k)^n dx = \frac{1}{n + 1} (x + k)^{n+1} + C$$

In our present case, the constants are set in the following way:

$$k = -1 \text{ and } n = -2$$

So that our result can be written down as follows:

$$\frac{1}{-1} (u - 1)^{-1} + C$$

Putting what was intended to be a form which shows clearly how the rule of integration has been applied into a rather neater form, we have as our first result:

$$I_6 = -\frac{1}{u - 1} + C$$

Our next integral is:

$$I_7 = \int \frac{2}{u - 1} du$$

This is, of course, equivalent to:

$$2 \int \frac{1}{u-1} du$$

and we can apply our favourite rule of integration for dealing with partial fractions expansions to write down the following solution:

$$I_7 = 2 \ln|u-1| + C$$

The third integral in this group is the following:

$$I_8 = \int \frac{2u+3}{u^2+u+1} du$$

This is another integral of the same form as I_5 from the first group, and we can apply exactly the same standard integral which proved useful in that particular case. Here, however, the constants in the standard integral take the following values:

$$p = 2, q = 3, b = 1 \text{ and } c = 1$$

The coefficients in the standard integral's solution which are rational formulae now take the following values:

$$\begin{aligned} \frac{bp-2q}{\sqrt{4c-b^2}} &= \frac{1 \cdot 2 - 2 \cdot 3}{\sqrt{4 \cdot 1 - 1^2}} = -\frac{4}{\sqrt{3}} \\ \frac{2}{\sqrt{4c-b^2}} &= \frac{2}{\sqrt{4 \cdot 1 - 1^2}} = \frac{2}{\sqrt{3}} \end{aligned}$$

(We note that the second of these is exactly the same as it was in the case of the solution to I_5 .) We are now ready to write down our solution as follows:

$$\ln|u^2 + u + 1| + \frac{4}{\sqrt{3}} \tan^{-1} \frac{2}{\sqrt{3}} \left(u + \frac{1}{2}\right) + C$$

As we did with the solution of I_5 earlier, we can multiply out the argument of the inverse tangent to show our solution as follows:

$$I_8 = \ln|u^2 + u + 1| + \frac{4}{\sqrt{3}} \tan^{-1} \left(\frac{2u+1}{\sqrt{3}}\right) + C$$

Example 6: Integrating I_9 - Evaluating J_2 , Part 2

We now come to the final intermediate integral, as well as the final component of the term of our solution in terms of u which we have named J_2 . This integral deserves its own separate subsection. The integral in question is this one:

$$I_9 = \int \frac{3u + 3}{(u^2 + u + 1)^2} du$$

This is a more challenging integral than any of the others we have had to face in this example – indeed, we have not encountered an integral quite like this one, containing as it does a squared irreducible quadratic in the denominator. However, we need not be defeated. We will start by taking the common coefficient of 3 outside the integral sign, as follows:

$$3 \int \frac{u + 1}{(u^2 + u + 1)^2} du$$

The remaining integral would be easy to deal with if only the numerator were $2u + 1$ - this is the derivative of the quadratic which appears in the denominator, and its presence in the numerator would give us a function-derivative pair, allowing us to generate a solution very quickly using the formulaic rule of integration derived from basic u -substitution.

Well, we could use a little crafty algebra to split up the integrand so that at least part of it satisfies this requirement – the remaining part would need further work, however. The algebra we would need to do is to add and subtract u from the numerator. This, of course, would leave the overall value of the numerator unchanged, but would let us split the integrand up into two separate rational expressions. Here is what would happen to the numerator:

$$u + 1 + u - u = 2u + 1 - u$$

We could then split the integrand into two parts as follows:

$$\frac{u + 1}{(u^2 + u + 1)^2} = \frac{2u + 1 - u}{(u^2 + u + 1)^2} = \frac{2u + 1}{(u^2 + u + 1)^2} - \frac{u}{(u^2 + u + 1)^2}$$

This would certainly allow us to integrate the first part with ease, but the second part looks no easier than the integrand we started with, so this may not be the best approach after all.

Instead, let's try the approach we took in [Part 1 Section 12](#) when we were dealing with integrands containing an irreducible quadratic in the denominator, sometimes under a square root sign. The current integrand may not prove so different, since the quadratic in our denominator here is squared – in other words, just raised to a different power.

Our first step in [Part 1](#) was to complete the square for the quadratic in the denominator – we can do this using the formula derived in [Appendix 3 Section 1](#) or simply “by inspection” to obtain the following equivalence:

$$u^2 + u + 1 = \left(u + \frac{1}{2}\right)^2 + \frac{3}{4}$$

This now allows us to use the substitution:

$$v = u + \frac{1}{2} \text{ so that } du = dv$$

We must note further that the numerator of our rational expression in u can be re-expressed using our substitution formula as follows:

$$u + 1 = v - \frac{1}{2} + 1 = v + \frac{1}{2}$$

Applying the substitution to our integral in u , we now have the following integral in terms of v , which we will give for reference the name I_v :

$$I_v = \int \frac{v + \frac{1}{2}}{\left(v^2 + \frac{3}{4}\right)^2} dv$$

which we can immediately split into two separate integrals as follows:

$$I_{10} = \int \frac{v}{\left(v^2 + \frac{3}{4}\right)^2} dv \text{ and } I_{11} = \int \frac{\frac{1}{2}}{\left(v^2 + \frac{3}{4}\right)^2} dv = \frac{1}{2} \int \frac{1}{\left(v^2 + \frac{3}{4}\right)^2} dv$$

so that our complete solution for I_v will be found using the following formula:

$$I_v = I_{10} + I_{11}$$

The first integral, I_{10} , contains a function-derivative pair with a coefficient – we note that the derivative of the bracketed expression in the denominator is $2v$ – which is the expression in the numerator multiplied by 2. It has the form:

$$kg'(f(v))f'(v)$$

where:

$$g'(f(v)) = \frac{1}{\left(v^2 + \frac{3}{4}\right)^2}; g'(v) = \frac{1}{v^2} = v^{-2} \text{ so that } g(v) = -v^{-1} = -\frac{1}{v}$$

$$f(v) = v^2 + \frac{3}{4}; f'(v) = 2v \text{ and } k = \frac{1}{2}$$

Our solution is therefore obtained from the formula:

$$kg(f(v)) + C$$

which in this case represents:

$$-\frac{1}{2} \cdot \frac{1}{v^2 + \frac{3}{4}} + C \text{ or, multiplying out, } -\frac{1}{2v^2 + \frac{3}{2}} + C$$

Multiplying above and below the line by 2, we can remove the fractional term, in order to arrive at this neater solution:

$$-\frac{2}{4v^2 + 3} + C$$

Our second integral in terms of v , I_{11} , with the coefficient of $1/2$ removed, is as follows:

$$I_{10} = \int \frac{1}{\left(v^2 + \frac{3}{4}\right)^2} dv$$

As we did in [Part 1 Section 12](#), we will handle this using tangent-substitution, making the following substitution:

$$v = \frac{\sqrt{3}}{2} \tan w \text{ so that } \frac{dv}{dw} = \frac{\sqrt{3}}{2} \sec^2 w \text{ and } dv = \frac{\sqrt{3}}{2} \sec^2 w dw$$

Plugging in these substitutions, we now have the following integral in terms of w :

$$\int \frac{1}{\left(\frac{3}{4} \tan^2 w + \frac{3}{4}\right)^2} \cdot \frac{\sqrt{3}}{2} \sec^2 w dw$$

Using the trigonometric formula:

$$\tan^2 x + 1 = \sec^2 x$$

the bracketed expression in the denominator reduces to:

$$\frac{3}{4}(\tan^2 w + 1) = \frac{3}{4} \sec^2 w$$

So that the whole integral becomes:

$$\int \frac{\frac{\sqrt{3}}{2} \sec^2 w}{\frac{9}{16} \sec^4 w} dw = \frac{\sqrt{3}}{2} \cdot \frac{16}{9} \int \frac{1}{\sec^2 w} dw = \frac{8}{3\sqrt{3}} \int \frac{1}{\sec^2 w} dw = \frac{8}{3\sqrt{3}} \int \cos^2 w dw$$

We can probably feel comfortable by this stage of the book about treating the final right-hand side integral as a standard integral, which we can quote from [Appendix 4 Section 2](#) as follows:

$$\int \cos^2 x dx = \frac{1}{2} \left(x + \frac{1}{2} \sin 2x \right) + C$$

This lets us write out our solution to the above integral as:

$$\frac{4}{3\sqrt{3}} \left(w + \frac{1}{2} \sin 2w \right) + C$$

We must now recall our trigonometric substitution, which was:

$$v = \frac{\sqrt{3}}{2} \tan w \text{ so that } w = \tan^{-1} \frac{2}{\sqrt{3}} v$$

Reversing this substitution, our solution in terms of v becomes:

$$\frac{4}{3\sqrt{3}} \left(\tan^{-1} \frac{2}{\sqrt{3}} v + \frac{1}{2} \sin 2 \tan^{-1} \frac{2}{\sqrt{3}} v \right) + C$$

Recalling that we have met compositions of trigonometric and inverse trigonometric functions several times before in this book, and that we have found that they are reducible to expressions which do not themselves involve any trigonometric functions, we must now digress briefly to perform the necessary reduction on the expression:

$$\sin 2 \tan^{-1} x$$

which is the form taken by the second term inside the brackets in the above first-draft solution.

We can simplify this using another of the series of results which deal with this type of situation, of which we have encountered several already. The one we need here (for which I have given a full derivation in [Appendix 4 Section 4](#)) is as follows:

$$\sin 2 \tan^{-1} x = \frac{2x}{x^2 + 1}$$

The term in our integral solution in terms of v which we wanted to simplify is therefore as follows:

$$\frac{1}{2} \sin 2 \tan^{-1} \frac{2}{\sqrt{3}} v = \frac{1}{2} \cdot \frac{\frac{2\sqrt{3}}{\sqrt{3}} v}{\frac{4}{3} v^2 + 1} = \frac{1}{2} \cdot \frac{4}{\sqrt{3}} \cdot \frac{v}{\frac{4}{3} v^2 + 1} = \frac{2}{\sqrt{3}} \cdot \frac{v}{\frac{4}{3} v^2 + 1}$$

Multiplying the final right-hand side above and below the line by 3 turns this into:

$$\frac{2}{\sqrt{3}} \cdot \frac{3v}{4v^2 + 3} = \frac{2\sqrt{3}v}{4v^2 + 3}$$

Our draft solution in terms of v is now as follows:

$$\frac{4}{3\sqrt{3}} \left(\tan^{-1} \frac{2}{\sqrt{3}} v + \frac{2\sqrt{3}v}{4v^2 + 3} \right) + C$$

And when we reapply the coefficient of $1/2$ which we took outside the integral sign before starting off down this convoluted route, we have the following solution in terms of v for the integral I_{11} :

$$I_{11} = \frac{1}{2} \cdot \frac{4}{3\sqrt{3}} \left(\tan^{-1} \frac{2}{\sqrt{3}} v + \frac{2\sqrt{3}v}{4v^2 + 3} \right) + C = \frac{2}{3\sqrt{3}} \left(\tan^{-1} \frac{2}{\sqrt{3}} v + \frac{2\sqrt{3}v}{4v^2 + 3} \right) + C$$

Before we put the two parts of our solution to I_v together, it might be convenient to multiply out the brackets in the right-hand side above, to obtain this:

$$\frac{2}{3\sqrt{3}} \tan^{-1} \frac{2}{\sqrt{3}} v + \frac{2}{3\sqrt{3}} \cdot \frac{2\sqrt{3}v}{4v^2 + 3} + C = \frac{2}{3\sqrt{3}} \tan^{-1} \frac{2}{\sqrt{3}} v + \frac{4v}{12v^2 + 9} + C$$

We must now put the two solutions in terms of v which we have just laboured so hard to obtain using the formula we set out above for the overall integral in terms of v , which was:

$$I_v = I_{10} + I_{11}$$

Remembering that the solution to I_{10} was found to have an opening negative sign, we will reverse the order of the two solutions to combine them as follows:

$$\frac{2}{3\sqrt{3}} \tan^{-1} \frac{2}{\sqrt{3}} v + \frac{4v}{12v^2 + 9} - \frac{2}{4v^2 + 3} + C$$

The last two terms can be expressed with the common denominator $12v^2 + 9$, (which is, of course, $4v^2 + 3$ multiplied by 3) giving the following result in terms of v :

$$I_v = \frac{2}{3\sqrt{3}} \tan^{-1} \frac{2}{\sqrt{3}} v + \frac{4v - 6}{12v^2 + 9} + C$$

We have taken all this trouble to solve the intermediate integral I_9 ; in order to display the final solution to this integral in terms of u , we must reverse our first substitution, which we recall was:

$$v = u + \frac{1}{2}$$

The above solution in v this transforms into the following solution in u :

$$\frac{2}{3\sqrt{3}} \tan^{-1} \frac{2}{\sqrt{3}} \left(u + \frac{1}{2} \right) + \frac{4 \left(u + \frac{1}{2} \right) - 6}{12 \left(u + \frac{1}{2} \right)^2 + 9} + C$$

The argument of the inverse tangent function can be multiplied out and simplified a little as follows:

$$\frac{2}{\sqrt{3}} \left(u + \frac{1}{2} \right) = \frac{2u}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{2u + 1}{\sqrt{3}}$$

The second, rational, term expands to give the following result:

$$\frac{4u + 2 - 6}{12\left(u^2 + u + \frac{1}{4}\right) + 9} = \frac{4u - 4}{12u^2 + 12u + 3 + 9} = \frac{4u - 4}{12u^2 + 12u + 12}$$

We can cancel out the common factor of 4 top and bottom, to get to the following:

$$\frac{u - 1}{3u^2 + 3u + 3} = \frac{u - 1}{3(u^2 + u + 1)}$$

Our preliminary solution in u is now tidied up a little to look as follows:

$$\frac{2}{3\sqrt{3}} \tan^{-1}\left(\frac{2u + 1}{\sqrt{3}}\right) + \frac{u - 1}{3(u^2 + u + 1)} + C$$

Lastly, we must recall that we moved a coefficient of 3 outside the integral sign when we first began looking at the intermediate integral I_9 . This coefficient must now be restored, at which point it will cancel with the coefficients present in the denominators in the solution shown above, to give the following final solution in terms of u :

$$I_9 = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2u + 1}{\sqrt{3}}\right) + \frac{u - 1}{u^2 + u + 1} + C$$

Example 6: Putting the Intermediate Integrals Together to Construct J_2

We have now evaluated all four of the intermediate integrals I_6 to I_9 which form the components of the part of our final solution which we have abbreviated under the name J_2 . It will be useful to list all these solutions together here so that the mechanics of combining them together will be clearer. These solutions are as follows:

$$I_6 = -\frac{1}{u - 1} + C$$

$$I_7 = 2 \ln|u - 1| + C$$

$$I_8 = \ln|u^2 + u + 1| + \frac{4}{\sqrt{3}} \tan^{-1}\left(\frac{2u + 1}{\sqrt{3}}\right) + C$$

$$I_9 = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2u + 1}{\sqrt{3}}\right) + \frac{u - 1}{u^2 + u + 1} + C$$

The formula which will combine these to provide us with J_2 is:

$$J_2 = \frac{2}{3} (I_6 - I_7 + I_8 + I_9)$$

Since the solution to I_6 has a negative sign, we will move both it and I_7 , which is provided with its own negative sign by the formula, to the end; also, we will first add together I_8 and I_9 , both of which have inverse tangents in them which can conveniently be combined. The sum of I_8 and I_9 , temporarily omitting the constant of integration, is as follows:

$$\ln|u^2 + u + 1| + \frac{4}{\sqrt{3}} \tan^{-1}\left(\frac{2u+1}{\sqrt{3}}\right) + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2u+1}{\sqrt{3}}\right) + \frac{u-1}{u^2 + u + 1}$$

Adding together the inverse tangent terms gives us the following result for this preliminary stage:

$$\ln|u^2 + u + 1| + \frac{6}{\sqrt{3}} \tan^{-1}\left(\frac{2u+1}{\sqrt{3}}\right) + \frac{u-1}{u^2 + u + 1}$$

Since:

$$\frac{6}{\sqrt{3}} = \frac{2.3}{\sqrt{3}} = 2\sqrt{3}$$

we can rewrite this more neatly in this way:

$$\ln|u^2 + u + 1| + 2\sqrt{3} \tan^{-1}\left(\frac{2u+1}{\sqrt{3}}\right) + \frac{u-1}{u^2 + u + 1}$$

Adding I_6 and subtracting I_7 and applying the overall coefficient of $2/3$ brings us to the following content for J_2 :

$$\frac{2}{3} \left(\ln|u^2 + u + 1| + 2\sqrt{3} \tan^{-1}\left(\frac{2u+1}{\sqrt{3}}\right) + \frac{u-1}{u^2 + u + 1} - \frac{1}{u-1} - 2 \ln|u-1| \right)$$

(Since this is only an intermediate stage, I have once again omitted the constant of integration. It will be brought back at the end of the process.)

The difference of two rational expressions in the above solution can be combined into one single expression with the common denominator:

$$(u-1)(u^2 + u + 1)$$

With this denominator, the numerator of the combined expression will be:

$$(u-1)^2 - (u^2 + u + 1) = u^2 - 2u + 1 - u^2 - u - 1 = -3u$$

and having made this change and gathering the logarithmic terms together, the content of J_3 can now be stated as:

$$\frac{2}{3} \left(\ln|u^2 + u + 1| - 2 \ln|u-1| + 2\sqrt{3} \tan^{-1}\left(\frac{2u+1}{\sqrt{3}}\right) - \frac{3u}{(u-1)(u^2 + u + 1)} \right)$$

It will help us further to recall at this stage that:

$$(u - 1)(u^2 + u + 1) = u^3 - 1$$

indeed, this is how we initially expanded out $u^3 - 1$ in order to allow us to use the Method of Partial Fractions! We can therefore tidy up our expression for J_2 even further in the following way:

$$J_2 = \frac{2}{3} \left(\ln|u^2 + u + 1| - 2 \ln|u - 1| + 2\sqrt{3} \tan^{-1} \left(\frac{2u + 1}{\sqrt{3}} \right) - \frac{3u}{u^3 - 1} \right)$$

Example 6: Combining the Components to Obtain I_u

It will make the task of subtracting the above result for J_2 from the earlier result we obtained for J_1 by multiplying out the brackets to obtain this version:

$$J_2 = \frac{2}{3} \ln|u^2 + u + 1| - \frac{4}{3} \ln|u - 1| + \frac{4\sqrt{3}}{3} \tan^{-1} \left(\frac{2u + 1}{\sqrt{3}} \right) - \frac{2u}{u^3 - 1}$$

We should now recall that we have already obtained the following expression for the component of our solution, J_1 , and it was as follows:

$$J_1 = \ln|u^2 + u + 1| + 2\sqrt{3} \tan^{-1} \left(\frac{2u + 1}{\sqrt{3}} \right) - 2 \ln|u - 1|$$

The solution in terms of u will be obtained by applying the following formula:

$$I_u = J_1 - J_2$$

Carrying out the subtraction term by term, we will obtain the following solution for I_u :

$$\frac{1}{3} \ln|u^2 + u + 1| - \frac{2}{3} \ln|u - 1| + \frac{2\sqrt{3}}{3} \tan^{-1} \left(\frac{2u + 1}{\sqrt{3}} \right) + \frac{2u}{u^3 - 1} + C$$

As a final step, we might like to simplify the coefficient of the inverse tangent term in the following way:

$$\frac{2\sqrt{3}}{3} = \frac{2\sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{2}{\sqrt{3}}$$

This brings us to the final expression of our overall solution in terms of u :

$$I_u = \frac{1}{3} \ln|u^2 + u + 1| - \frac{2}{3} \ln|u - 1| + \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2u + 1}{\sqrt{3}} \right) + \frac{2u}{u^3 - 1} + C$$

Example 6: Reversing the Substitution

We have now arrived at the final stage! All that is left is to reverse the power-substitution we made right at the outset in order to take the cube root in our integrand out of consideration. A brief reminder is probably in order; the substitution was:

$$u^3 = \frac{x+1}{x-1} \text{ so that } u = \sqrt[3]{\frac{x+1}{x-1}}$$

It is probably apparent that when we reverse this substitution, because of the number of instances of the substitution variable u in our solution above, as well as the several different powers to which it is raised, we will find the cube root of our rational expression in x appearing frequently. In order to avoid burdening the page with a spattering of cube root signs - or even bracketed rational expressions raised to fractional powers, I will propose that we abbreviate the entire rational expression under the cube root, as follows:

$$X = \frac{x+1}{x-1}$$

This means that we can represent the various instances of the substitution variable u write neatly, replacing it and its various powers as follows:

$$u = X^{1/3}, u^2 = X^{2/3} \text{ and } u^3 = X$$

Using this abbreviated notation, we can easily state our final solution in terms of x (or in the current version in terms of X) as follows:

$$I = \frac{1}{3} \ln|X^{2/3} + X^{1/3} + 1| - \frac{2}{3} \ln|X^{1/3} - 1| + \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2X^{1/3} + 1}{\sqrt{3}} \right) + \frac{2X^{1/3}}{X-1} + C$$

where $X = \frac{x+1}{x-1}$

It is worth noting that we can make one further simplification of the final term in this solution, as long as we are prepared to mix instances of the abbreviation X with instances of the actual variable of integration x

We start by noting that, in its expanded form, the denominator of this final term is:

$$X - 1 = \frac{x+1}{x-1} - 1 = \frac{x+1}{x-1} - \frac{x-1}{x-1} = \frac{x+1-(x-1)}{x-1} = \frac{2}{x-1}$$

The final term as a whole can therefore be written as:

$$2X^{1/3} \cdot \frac{x-1}{2} = X^{1/3}(x-1)$$

This allows us to state an alternative final solution, which would be as follows:

$$I = \frac{1}{3} \ln|X^{2/3} + X^{1/3} + 1| - \frac{2}{3} \ln|X^{1/3} - 1| + \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2X^{1/3} + 1}{\sqrt{3}} \right) + X^{1/3}(x - 1) + C$$

where $X = \frac{x + 1}{x - 1}$

Example 6: A Concluding Summary

The evaluation of this integral has been such a complicated process that it would make sense to give a brief summary at the end of the process, showing how we got to it and the various different techniques we employed – the use of multiple techniques, after all, being the main subject of this section. Our procedure in evaluating the integral I was as follows:

1. We began by making a power-substitution to remove the cube root from our subsequent working. The substitution we chose required some algebraic manipulation in order to make it simpler for us to find our expression for dx .
2. We used some more algebraic manipulation to put the integrand which resulted from power-substitution into a two-term form which was amenable to partial fractions expansion so as to make it integrable.
3. We obtained partial fractions expansions for each of the two terms obtained at the previous stage.
4. We then integrated each of the resulting terms in the partial fractions expansions; all except the very last term in the second expansion could be integrated using rules we have already established. The awkward final term, which had a squared irreducible quadratic in its denominator, had to be integrated using the multi-stage process we developed in [Part 1 Section 12](#). This entailed completing the square for the quadratic in the denominator, and then making a simple substitution on the squared part of the resulting expression. The resulting integral was split into two terms, one of which was integrated using a standard rule, and the other using a trigonometric substitution.
5. The various components of the solution, obtained using a term-by-term approach, were then combined together and the result simplified to obtain our solution in terms of u .
6. Finally, we reversed the original power-substitution and expressed the final solution in terms of x using an abbreviation for the rational expression inside the integrand's cube root sign to simplify the presentation of our result.

This example must hold the record in this book for the number of different techniques deployed in its solution. It also serves to illustrate the importance of naming intermediate integrals appropriately to avoid becoming lost in complexity. A final remark might be added on the variety of different terms in the solution, which includes logarithmic terms, an inverse tangent and an algebraic expression.

9 Trigonometric Integrals

In this section, leaving behind our detailed look at a range of integrals containing rational algebraic expressions, we will turn to the question of integrating various combinations of trigonometric functions. The trigonometric functions are connected by a rich set of identities,

which allow us to start with one kind of expression containing these functions and transform it into another kind of expression, which may in some sense or other be more useful to us. When we are concerned with integrating trigonometric expressions, of course, the identities which will prove most useful to us are those where we start with an expression which we cannot easily integrate and transform it into one which we can integrate.

In this regard, the identities which are likely to be most helpful to us include the power-reduction formulae, which allow us to resolve squares of trigonometric functions into expressions without any powers in them. Two such useful formulae are as follows:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

While it doesn't actually help us get from something we can't integrate to something we can, the Pythagorean identity linking sine and cosine may be of use to us in allowing powers of either sine or cosine to be transformed into powers of the other of the two functions. The identity is, of course:

$$\sin^2 x + \cos^2 x = 1$$

and its various rearrangements may prove of use to us.

We do not have a power-reduction formula of precisely the same nature as the ones we listed for the squares of the sine and cosine for the square of the tangent function, but we do have an identity which transforms squared tangent functions into something different, which we can certainly integrate – this is a form of the identity connecting the tangent and secant functions, which is as follows:

$$\sec^2 x - \tan^2 x = 1$$

and the particular rearrangement which is likely to be most useful to us is this one:

$$\tan^2 x = \sec^2 x - 1$$

On the right-hand side, we have an instance of $\sec^2 x$, which, being just the derivative of $\tan x$, is easily integrated.

Another set of identities which will prove very useful to us are the product-to-sum formulae, which take products of trigonometric formulae and transform them into the sum, of two different trigonometric functions. These formulae we are derived from the addition formulae – a set of identities which are among the first to be learnt when familiarising ourselves with trigonometry. [Appendix 4 Section 3](#) has a brief discussion of their derivation, and I list here the product-to-sum formulae which will be useful to us:

$$\sin \alpha \sin \beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$$

$$\cos \alpha \cos \beta = \frac{1}{2}(\cos(\alpha - \beta) + \cos(\alpha + \beta))$$

$$\sin \alpha \cos \beta = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta))$$

I have listed three formulae here, rather than the four shown in [Appendix 4](#), since the last of those in the appendix is really only a restatement of the third using different placeholders α and β .

It may not be immediately clear how these product-to-sum formulae help us in our task of integration – if, however, we imagine the placeholders α and β to be multiples of x , for example as follows:

$$\alpha = 7x \text{ and } \beta = 5x \text{ so that } \alpha - \beta = 2x \text{ and } \alpha + \beta = 12x$$

then we can see that the first of the above formulae allows us to state the following:

$$\sin 7x \sin 5x = \frac{1}{2}(\cos 2x - \cos 12x)$$

This would be very helpful in integration, since the right-hand side, apart from the coefficient of $1/2$, which is no real obstacle to us, its contents are easy to integrate using standard rules.

We will now proceed to look at some examples, starting with my recommended approach to an integral which we attempted to deal with by parts in [Part 2 Section 12](#), but which failed to respond to that method.

Example 1: Integrating $\sin 3x \cos 5x$: the Recommended Method

In [Part 2 Section 12 Example 4](#), we tried to evaluate the following integral:

$$I = \int \sin 3x \cos 5x \, dx$$

using integration by parts. The technique did not work, giving us a useless tautological formula in terms of I . I pointed out in the notes following that example that such behaviour can be taken as pretty reliable evidence that we have made the wrong choice of integration technique, and that this particular integral can be evaluated quite simply with the aid of a trigonometric identity.

The identity we need is one of the product-to-sum formulae we discussed at the start of this section – the one we need in this instance is as follows:

$$\sin \alpha \cos \beta = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta))$$

Applying this identity to the expression which constitutes our integrand by setting $\alpha = 3x$ and $\beta = 5x$, we obtain the following:

$$\sin 3x \cos 5x = \frac{1}{2}(\sin 8x + \sin(-2x))$$

Simple consideration of the properties of the sine function (or even a quick visualisation of its graph) will confirm that $\sin(-\alpha) = -\sin \alpha$. We can therefore rewrite the right-hand side above as follows:

$$\frac{1}{2}(\sin 8x - \sin 2x)$$

In order to integrate this expression, we can take the coefficient of $1/2$ outside the integral sign and integrate the bracketed part of the expression term by term, as shown here:

$$I = \frac{1}{2} \left(\int \sin 8x \, dx - \int \sin 2x \, dx \right)$$

The right-hand side expression can be evaluated quite simply to give us a fairly speedy final solution of:

$$I = \frac{1}{2} \left(-\frac{1}{8} \cos 8x + \frac{1}{2} \cos 2x \right) + C = \frac{1}{4} \cos 2x - \frac{1}{16} \cos 8x + C$$

The final right-hand side is obtained by swapping the two terms in brackets in order to eliminate an ugly leading minus sign, and multiplying out the brackets to give a simple two-term expression.

Example 2

Here, we will complete a process which I started immediately before [Example 1](#) when explaining how the product-to-sum formulae could prove helpful tools in evaluating integrals of products of trigonometric functions. We will integrate the example expression I used there, and evaluate:

$$I = \int \sin 7x \sin 5x \, dx$$

In the explanation above, we established that the appropriate product-to-sum formula would allow us to restate the integrand as:

$$\frac{1}{2}(\cos 2x - \cos 12x)$$

so that our integral can be expressed in the following term-by-term version:

$$I = \frac{1}{2} \left(\int \cos 2x \, dx - \int \cos 12x \, dx \right)$$

This restated integral can be evaluated easily to give the following solution:

$$I = \frac{1}{2} \left(\frac{1}{2} \sin 2x - \frac{1}{12} \sin 12x \right) + C = \frac{1}{4} \sin 2x - \frac{1}{24} \sin 12x + C$$

Note

The process of applying the product-to-sum formulae as an aid to integration is so straightforward that no further examples may seem necessary. However, a couple of remarks are in order.

First, we can see that the very first integral we examined in [Part 1](#) of this book, which was:

$$I = \int \sin x \cos x \, dx$$

will actually respond quite easily to the use of the formulae we have been discussing. We can apply the last of the three formulae I listed at the start of this section, with the following values for the placeholders:

$$\alpha = x \text{ and } \beta = x$$

The integrand becomes:

$$\frac{1}{2} (\sin 2x + \sin 0) = \frac{1}{2} \sin 2x$$

since the properties of the sine function tell us that $\sin 0 = 0$. Our integral now becomes:

$$I = \frac{1}{2} \int \sin 2x \, dx = -\frac{1}{4} \cos 2x + C$$

Well, this looks nothing like the solution we obtained in [Part 1](#) using u -substitution – that result was:

$$\frac{1}{2} \sin^2 x + C$$

However, there is a way to reconcile these apparently different results obtained by the different methods. In order to see how this works, we first recall the power-reduction formula for the sine function, which states that:

$$\sin^2 x = \frac{1}{2} (1 - \cos 2x)$$

This means that the solution from [Part 1](#) can be rewritten as:

$$\frac{1}{4} (1 - \cos 2x) + C = \frac{1}{4} - \frac{1}{4} \cos 2x + C = -\frac{1}{4} \cos 2x + C' \text{ where } C' = C + \frac{1}{4}$$

We can now see that the two solutions are actually the same solution *subject to a difference in how we express the constant of integration*. We have encountered this phenomenon, where

two different methods of integration provide solutions which differ in this way – the difference here is that the relationship between the two alternative solutions is not at first apparent, until we notice the relationship between:

$$\sin^2 x \text{ and } -\frac{1}{4} \cos 2x$$

which actually only differ by a constant quantity.

My second remark is simply to point out that it is easy to derive three useful standard integrals from the use of the product-to-sum formulae. I set out the derivation of all three in [Appendix 5 Section 21](#), and they are listed for reference in [Appendix 4 Section 2](#).

Example 3

We will now move on from products of sines and cosines of multiples of x to look at integrands which contain products of powers of these two functions. We will start with:

$$I = \int \sin^2 x \cos^3 x dx$$

We will begin by making a restatement of the integrand which will help us to see how to proceed. We will extract $\cos x$ from the integrand and write it as a separate factor, as follows:

$$\sin^2 x \cos^2 x \cos x$$

We will then take advantage of the fact that we can now see a factor of $\cos^2 x$ to rewrite the integrand using a rearrangement of the Pythagorean identity, turning the integrand into the following expression:

$$\sin^2 x (1 - \sin^2 x) \cos x$$

At this point, we will be able to integrate by using the substitution:

$$u = \sin x \text{ so that } \frac{du}{dx} = \cos x \text{ and } dx = \frac{1}{\cos x} du \text{ or if we prefer } du = \cos x dx$$

This gives us the following integral in terms of u :

$$\int u^2 (1 - u^2) du = \int (u^2 - u^4) du$$

which can be integrated easily term-by-term to give this solution in terms of u :

$$\frac{1}{3} u^3 - \frac{1}{5} u^5 + C$$

Reversing the substitution - which is a very simple matter since, we must recall, this was a straightforward u -substitution, and not a trigonometric substitution where the substitution formula was expressed with x as the subject - we have the following solution in terms of x :

$$I = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$$

Example 3: A Quicker Solution

I have carried out the integration of [Example 3](#) above by setting out the process of u -substitution explicitly. It will be recalled, however, that we can short-circuit this procedure when the substitution is a straightforward one expressed with u as the subject.

In order to demonstrate this, we should observe that, once we have rearranged the integrand using some algebraic and trigonometric manipulation into the following form:

$$\sin^2 x (1 - \sin^2 x) \cos x$$

we find a function-derivative pair to be present. In other words, the integrand has the classic form:

$$kg'(f(x))f'(x)$$

where:

$$k = 1, g'(f(x)) = \sin^2 x (1 - \sin^2 x), f(x) = \sin x \text{ and } f'(x) = \cos x$$

$$g'(x) = x^2(1 - x^2) = x^2 - x^4 \text{ so that } g(x) = \frac{1}{3}x^3 - \frac{1}{5}x^5$$

Our rule of integration derived from u -substitution predicts a solution of the form:

$$kg(f(x)) + C = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$$

which is the same solution obtained from the explicit integration by substitution above.

Example 4

We will now integrate a similar expression to the one we have just dealt with. Our integral this time will be:

$$I = \int \sin^3 x \cos^8 x dx$$

In this case, we will extract a factor of $\sin x$ from the integrand to rewrite it as follows:

$$\sin^2 x \cos^8 x \sin x$$

We can then use a rearrangement of the Pythagorean identity to rewrite the integrand once more as follows:

$$(1 - \cos^2 x) \cos^8 x \sin x$$

We are now in a position to substitute in a broadly similar way to the approach we took with the previous example; we will make this substitution:

$$u = \cos x \text{ so that } \frac{du}{dx} = -\sin x \text{ and } dx = -\frac{1}{\sin x} du \text{ or if we prefer } du = -\sin x \, dx$$

This provides us with the following integral in terms of u :

$$\int -(1 - u^2)u^8 du = \int (u^2 - 1)u^8 du = \int (u^{10} - u^8) du$$

We move from the initial left-hand side above to the next stage in the chain of equalities by using the simple algebraic rule:

$$-(a - b) = b - a$$

to do away with the opening minus sign.

As before, this integrates very simply term by term to give this solution in terms of u :

$$\frac{1}{11}u^{11} - \frac{1}{9}u^9 + C$$

and when we finally reverse the substitution, we will have this solution in terms of x :

$$I = \frac{1}{11}\cos^{11}x - \frac{1}{9}\cos^9x + C$$

This solution has proceeded in an almost identical manner to the solution to [Example 3](#) above, apart from the introduction of a minus sign as a result of the differentiation which followed from choosing to substitute $u = \cos x$ instead of $u = \sin x$.

Example 4: Another Quicker Solution

As was the case with [Example 3](#), we can observe that once we have rearranged the integrand in the current example into the following form:

$$(1 - \cos^2 x) \cos^8 x \sin x$$

we will again be able to spot a function-derivative pair, but this time with a “hidden” constant coefficient. As we usually do in these cases, we represent the integrand in the form:

$$k g'(f(x)) f'(x)$$

In the current case, we have the following equivalences:

$g''f(x)) = (1 - \cos^2 x)\cos^8, f(x) = \cos x$ and $f'(x) = -\sin x$, requiring $k = -1$

$$g'(x) = (1 - x^2)x^8 = x^8 - x^{10} \text{ so that } g(x) = \frac{1}{9}x^9 - \frac{1}{11}x^{11}$$

We should note that we have to introduce a constant coefficient of -1 into our representation, since the derivative of $\cos x$ is $-\sin x$, and its negative sign must be neutralised since the original integrand has no negative sign.

Our rule of integration based on the method of substitution predicts a solution as follows:

$$kg(f(x)) + C = -\left(\frac{1}{9}\cos^9 x - \frac{1}{11}\cos^{11} x\right) + C = \frac{1}{11}\cos^{11} x - \frac{1}{9}\cos^9 x + C$$

and we can see that the final right-hand side is, of course, the same solution as the one we obtained by explicit integration by substitution.

Example 5

Both of the previous examples have been in a sense “convenient” examples, in that, once we had extracted a factor of either $\cos x$ or $\sin x$, we were left with an instance of the same trigonometric function squared, so that we could use the Pythagorean identity to transform this into an expression using the other function in that identity, paving the way for an easy substitution. This happened in each case because each integrand contained a factor of either $\cos x$ or $\sin x$ *cubed* to begin with

We must therefore investigate whether the technique we have been using in the last two examples can also be applied when $\cos x$ or $\sin x$ are raised to powers higher than 3. In this spirit, we will attempt a similar approach to the following integral:

$$I = \int \sin^2 x \cos^5 x dx$$

We will start, as before, moving a factor of $\cos x$ into an exposed position in the integrand by rewriting it as follows:

$$\sin^2 x \cos^4 x \cos x$$

It is not so clear how we can go about converting the remaining factor of $\cos^4 x$ into an expression involving only sines (so as to be able to substitute $u = \sin x$ as we did in [Example 3](#)) until we rewrite the new version of the integrand as follows:

$$\sin^2 x (\cos^2 x)^2 \cos x = \sin^2 x (1 - \sin^2 x)^2 \cos x$$

We can multiply out the squared bracketed factor to get the following revised integrand:

$$\sin^2 x (1 - 2\sin^2 x + \sin^4 x) \cos x$$

We are now able to substitute just as we did in [Example 3](#), using:

$$u = \sin x \text{ so that } du = \cos x \, dx$$

(I have here leapt straight to an expression in terms of differentials – the fuller reasoning is, of course, given in [Example 3](#) above; I have also chosen an expression for du in terms of dx , as distinct from my normal practice in this book since it may serve to make clearer the reason for pulling out a single factor of $\cos x$ in the way I have done.)

With this substitution, we can move straight to the following integral in terms of u :

$$\int u^2(1 - 2u^2 + u^4)du = \int (u^2 - 2u^4 + u^6)du$$

which, once again, integrates easily term by term to give us this solution in terms of u :

$$\frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$$

and we can very easily reverse our substitution to produce this final solution in terms of x :

$$I = \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C$$

By this time, it will I hope be obvious that we could use the rule of integration we derived from u -substitution just as easily here once we have got the integrand into the correct shape, and I hope there is no need to exhibit that method again.

Example 6

In this example, we will look at an integral which seems at first entirely similar to those preceding, but which will in the event serve to illustrate a very important point about the method we have been using in [Examples 3 to 5](#) above. Our integral will be as follows:

$$I = \int \sin^2 x \cos^4 x \, dx$$

We will notice right at the outset that both trigonometric functions in the integrand are raised to a even power – in the preceding three examples, at least one odd power has been present. That said, let's see if our method will still work.

We seem to have a choice – we can extract a cosine and aim to substitute $u = \sin x$ or extract a sine and substitute $u = \cos x$. Let's start with the first of these two options – the integrand is rewritten as:

$$\sin^2 x \cos^3 x \cos x$$

If we were to proceed with the substitution of $u = \sin x$, we would need to express the cubed $\cos x$ term in terms of $\sin x$, and we don't have a ready way of achieving that.

So let's try the other option, of starting by extracting a sine from the current factors. This gives us a rewritten integrand as follows:

$$\sin x \cos^3 x \sin x$$

The plan to go on and substitute $u = \cos x$ would then leave us with the necessity of expressing $\sin x$ in terms of $\cos x$. This is, of course, possible, but perhaps not helpful – we can use the Pythagorean identity to write:

$$\sin x = \sqrt{1 - \cos^2 x}$$

Bearing in mind that the substitution $u = \cos x$ gives us the equivalence $du = -\sin x \, dx$, we would be left with the following integral in terms of u :

$$-\int (1 - u^2)^{1/2} u^3 \, du$$

and this is really no easier than the integral in x with which we started out.

These two blind alleys should convince us that an alternative approach is necessary. We can see why it is made necessary by observing that our previous technique worked because at least one of the trigonometric functions in our earlier examples was raised to an odd power. This means that extracting one instance of such a function would leave an even power behind, which we can express fairly easily in terms of the other trigonometric function using the Pythagorean identity.

So can we rewrite the present integrand in such a way as to allow us to integrate it? Rather than applying the Pythagorean identity directly to the integrand, which would leave us an expression in only one of the two trigonometric functions (this would make it difficult to substitute u for the remaining function, since this would yield an expression for dx or du in terms of the *other* function!). Another tool we have at our disposal would be the power-reduction formulae, which allow us to rewrite the squares of each of the functions in our integrand in a simpler form.

Using these on the integrand, we can rewrite it as follows:

$$\sin^2 x \cos^4 x = \frac{1}{2}(1 - \cos 2x) \left(\frac{1}{2}(1 + \cos 2x) \right)^2$$

The right-hand side can be simplified as follows:

$$\frac{1}{2}(1 - \cos 2x) \cdot \frac{1}{4}(1 + \cos 2x)^2 = \frac{1}{8}(1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x)$$

The right-hand side multiplies out to give the following:

$$\frac{1}{8}(1 + 2\cos 2x + \cos^2 2x - \cos 2x - 2\cos^2 2x - \cos^3 2x)$$

If we gather like terms together, we will have the following version:

$$\frac{1}{8}(1 + \cos 2x - \cos^2 2x - \cos^3 2x)$$

The powers of $\cos 2x$ greater than 1 can be rewritten in the following way:

$$\cos^2 2x = \frac{1}{2}(1 + \cos 4x)$$

$$\cos^3 2x = \cos^2 2x \cos 2x = (1 - \sin^2 2x) \cos 2x$$

The reasoning in the second line above makes use of the Pythagorean identity to rewrite $\cos^2 2x$ in terms of $\sin^2 2x$.

We should immediately spot that the final right-hand side of the second line contains a function-derivative pair, so that it would be amenable to integration by substitution. We therefore need go no further in attempting to simplify it.

We should now put these simplifications back into the earlier expression for our integrand, so as to obtain the following

$$\frac{1}{8}(1 + \cos 2x - \frac{1}{2}(1 + \cos 4x) - (1 - \sin^2 2x) \cos 2x)$$

Multiplying out the term containing $\cos 4x$ and gathering the constant terms together gives us:

$$\frac{1}{8}\left(\frac{1}{2} + \cos 2x - \frac{1}{2}\cos 4x - (1 - \sin^2 2x) \cos 2x\right)$$

We can put this into a neater form ready for term-by-term integration by factorising out $1/2$ (and introducing a coefficient of 2 for those terms which are not multiplied by $1/2$). With this step, we have the following version of our integrand:

$$\frac{1}{16}(1 + 2 \cos 2x - \cos 4x - 2(1 - \sin^2 2x) \cos 2x)$$

This can now be integrated, by moving the coefficient of $1/16$ outside the integral sign and dealing with the rest of the expression term by term. We can integrate the first two terms inside the brackets with ease, and if we define the intermediate integral I_1 as follows:

$$I_1 = \int 2(1 - \sin^2 2x) \cos 2x \, dx$$

we can write out a preliminary sketch of our solution as follows:

$$\frac{1}{16}\left(x + \sin 2x - \frac{1}{4}\sin 4x - I_1\right)$$

We will now complete the job by evaluating the intermediate integral I_1 . I mentioned earlier that this contains a function-derivative pair, allowing the use of substitution. Accordingly, we will substitute:

$$u = \sin 2x \text{ so that } \frac{du}{dx} = 2 \cos 2x \text{ and } du = 2 \cos 2x dx$$

Our integral in terms of u is now quite a simple one:

$$\int (1 - u^2) du$$

which integrates term by term to produce this solution in terms of u :

$$u - \frac{1}{3}u^3 + C$$

and after reversing the substitution, this is our solution in terms of x :

$$\sin 2x - \frac{1}{3}\sin^3 2x + C$$

Putting this back into our earlier partial solution, we have the following expression for our solution to the original integral I :

$$\frac{1}{16} \left(x + \sin 2x - \frac{1}{4} \sin 4x - \sin 2x + \frac{1}{3} \sin^3 2x \right) + C$$

The terms in $\sin 2x$ can be gathered together to cancel each other out, so that we have the following neater version:

$$\frac{1}{16} \left(x - \frac{1}{4} \sin 4x + \frac{1}{3} \sin^3 2x \right) + C$$

We may prefer to multiply out the brackets, so that we end with a series of terms as follows:

$$I = \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C$$

Let's mention a possible restatement of this result, using the double-angle identity for the sine to write:

$$\sin 4x = 2 \sin 2x \cos 2x$$

This lets us give the following alternative version of our solution:

$$I = \frac{1}{16}x - \frac{1}{32}\sin 2x \cos 2x + \frac{1}{48}\sin^3 2x + C$$

where all the trigonometric functions are expressed as functions of $2x$.

We immediately notice that this solution is quite unlike those we obtained for earlier integrals of products of powers of trigonometric functions. The earlier solutions were all series of powers of a trigonometric function, while this last one – where the powers in the integrand were both even – has a term in x followed by terms in the sine (and cosine if we wish) of multiples of x .

Example 7

In order to get a better feel for the procedure we have had to adopt in this case, let's consider an example which contains some higher powers. Our integral here will be:

$$I = \int \sin^4 x \cos^6 x dx$$

We will try the approach we took in the previous example, where we rewrote the integrand in terms of powers of the square of the sine and cosine, which then allowed us to apply the power-reduction formulae. In this case, our integrand becomes:

$$(\sin^2 x)^2 (\cos^2 x)^3$$

Using the power-reduction formulae to rewrite the bracketed items in terms of $\cos 2x$ will give us the following expression:

$$\left(\frac{1}{2}(1 - \cos 2x)\right)^2 \left(\frac{1}{2}(1 + \cos 2x)\right)^3$$

We will immediately deal with the constant coefficients and take the result of multiplying them together out as a single coefficient. The operation on the coefficients is as follows:

$$\left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 = \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$$

so that our overall expression can now be written as:

$$\frac{1}{32} (1 - \cos 2x)^2 (1 + \cos 2x)^3$$

We now expand out the squared and cubed factors (using the Binomial Theorem for the cubed factor if you don't know the standard expansion) to obtain this expression:

$$\frac{1}{32} (1 - 2 \cos 2x + \cos^2 2x) (1 + 3 \cos 2x + 3 \cos^2 2x + \cos^3 2x)$$

In multiplying out this expression, we can multiply the second bracketed factor successively by each term in the first bracketed factor – this gives us the following three expressions, which we can use to gather together like terms:

$$\begin{aligned} & 1 + 3 \cos 2x + 3 \cos^2 2x + \cos^3 2x \\ & - 2 \cos 2x - 6 \cos^2 2x - 6 \cos^3 2x - 2 \cos^4 2x \end{aligned}$$

$$\cos^2 2x + 3\cos^3 2x + 3\cos^4 2x + \cos^5 2x$$

We can now proceed to gather together like terms from the above three expressions and write down the result, not forgetting the coefficient of $1/32$ which must be applied:

$$\frac{1}{32}(1 + \cos 2x - 2\cos^2 2x - 2\cos^3 2x + \cos^4 2x + \cos^5 2x)$$

so that our integral can now be restated as follows:

$$\frac{1}{32} \int (1 + \cos 2x - 2\cos^2 2x - 2\cos^3 2x + \cos^4 2x + \cos^5 2x) dx$$

The integrand is a series of powers of $\cos 2x$. We might decide, as we did in the previous example, to use trigonometric identities to simplify the integrand further. However, we could instead appeal to the standard integrals for powers of $\cos x$ listed in [Appendix 4 Section 2](#) (where they are justified on a basis of repeated use of the reduction formula for the integral of $\cos^n x$) together with the following useful rule of integration derived from the method of substitution:

$$\int f(kx) dx = \frac{1}{k} F(kx) + C$$

to integrate the expression term by term. This will almost certainly be a quicker and less error-prone procedure, so that's what we will do.

Applying the stated methods, the following table shows how each term of the integrand integrates to contribute the following material to our final solution – the third column shows the sign which is to be applied to each contribution as determined by the signs present in the integrand:

<i>Term</i>	<i>Contribution to Integral</i>	<i>Sign</i>
1	x	+
$\cos 2x$	$\frac{1}{2} \sin 2x$	+
$2\cos^2 2x$	$x + \frac{1}{2} \sin 2x \cos 2x$	–
$2\cos^3 2x$	$\frac{2}{3} \sin 2x + \frac{1}{3} \sin 2x \cos^2 2x$	–
$\cos^4 2x$	$\frac{3}{8} x + \frac{3}{16} \sin 2x \cos 2x + \frac{1}{8} \sin 2x \cos^3 2x$	+
$\cos^5 2x$	$\frac{8}{30} \sin 2x + \frac{4}{30} \sin 2x \cos^2 2x + \frac{1}{10} \sin 2x \cos^4 2x$	+

We can now gather together like terms, making sure we apply the correct sign as shown in the third column. I will gloss over in an airy fashion the fairly serious fraction arithmetic which this operation calls for (after all, this is not a book on arithmetic!), moving straight to the conclusion that our result will be as follows:

$$\begin{aligned} \frac{3}{8}x + \frac{1}{10}\sin 2x - \frac{5}{16}\sin 2x \cos 2x - \frac{2}{15}\sin 2x \cos^2 2x + \frac{1}{8}\sin 2x \cos^3 2x \\ + \frac{1}{10}\sin 2x \cos^4 2x \end{aligned}$$

We must finally be sure to reapply the coefficient of $1/32$ which we moved outside the integral sign. Bringing this back gives us the following final result:

$$\begin{aligned} I = \frac{3}{256}x + \frac{1}{320}\sin 2x - \frac{5}{512}\sin 2x \cos 2x - \frac{1}{240}\sin 2x \cos^2 2x + \frac{1}{256}\sin 2x \cos^3 2x \\ + \frac{1}{320}\sin 2x \cos^4 2x + C \end{aligned}$$

A Postscript to Example 6

It is worth a taking a moment to remark that in [Example 6](#), we did actually reduce the original integrand to a series of powers of $\cos 2x$, just as we did in [Example 7](#), but we then proceeded to expand these terms out further using trigonometric identities. For the satisfaction of those who consider that the approach of [Example 7](#) was more orderly and attractive (not to mention more practical in the case of the higher powers with which we started in [Example 7](#)), I will complete the integration of [Example 6](#) from the series of powers of $\cos 2x$ using the method of [Example 7](#).

By way of reminder, we reduced the integrand in [Example 6](#) to the following form:

$$\frac{1}{8}(1 + \cos 2x - \cos^2 2x - \cos^3 2x)$$

Instead of the further application of trigonometric identities, we can use this expression to rewrite our original integral as follows:

$$\frac{1}{8} \int (1 + \cos 2x - \cos^2 2x - \cos^3 2x) dx$$

As we did in [Example 7](#), this time we will set out a table showing the contribution made by each term inside the integral sign to the complete integral; we will again use the table of integrals of powers of $\cos x$ in conjunction with the rule of integration for integrands of the form $f(kx)$ to find the contribution made by each term. Here is the table:

Term	Contribution to Integral	Sign
1	x	+
$\cos 2x$	$\frac{1}{2}\sin 2x$	+
$\cos^2 2x$	$\frac{1}{2}x + \frac{1}{4}\sin 2x \cos 2x$	−
$\cos^3 2x$	$\frac{1}{3}\sin 2x + \frac{1}{6}\sin 2x \cos^2 2x$	−

Gathering together like terms gives the following result for the integral in the above expression (not, of course, the final result – we will still have a coefficient to apply):

$$\frac{1}{2}x + \frac{1}{6}\sin 2x - \frac{1}{4}\sin 2x \cos 2x - \frac{1}{6}\sin 2x \cos^2 2x$$

Applying the previously extracted coefficient of $1/8$ gives the following final solution to our original integral:

$$I = \frac{1}{16}x + \frac{1}{48}\sin 2x - \frac{1}{32}\sin 2x \cos 2x - \frac{1}{48}\sin 2x \cos^2 2x + C$$

We will recall, however, that the solution we obtained by repeated use of trigonometric identities in [Example 6](#) was as follows:

$$\frac{1}{16}x - \frac{1}{32}\sin 2x \cos 2x + \frac{1}{48}\sin^3 2x + C$$

Can these apparently different results be reconciled? In order to do so, we would have to demonstrate that:

$$\frac{1}{48}\sin 2x - \frac{1}{48}\sin 2x \cos^2 2x = \frac{1}{48}\sin^3 2x$$

since the remaining terms are identical in both solutions. This is actually not as difficult as it may at first appear – in particular, all terms on both sides have identical coefficients, so that we can reduce our problem to that of showing that:

$$\sin 2x - \sin 2x \cos^2 2x = \sin^3 2x$$

In order to demonstrate the equality, we will apply the Pythagorean identity to the $\cos^2 2x$ factor on the left-hand side. This will give us:

$$\sin 2x - \sin 2x (1 - \sin^2 2x) = \sin 2x - \sin 2x + \sin^3 2x = \sin^3 2x$$

and so, after cancelling out the $\sin 2x$ terms, our required equivalence is proved. Therefore, the two apparently different solutions are shown to be different ways of expressing the same solution.

This kind of phenomenon, where different integration techniques produce solutions which appear not to be equivalent, does arise when integrating expressions which contain trigonometric functions. The trigonometric functions and their powers are, of course, connected by a set of very specific identities, and these frequently have to be deployed to demonstrate the equivalence of solutions which appear to be quite different.

Summary for Powers of $\sin x$ and $\cos x$

In the last four examples, numbered 3 to 7, we have been evaluating integrals of the following form:

$$\int \sin^m x \cos^n x dx$$

We have used several different techniques, and we are now in a position to summarise these and give some pointers as to which can be applied and when.

- If either m or n is odd, we can use the following approaches:
 - In cases where m is odd, we can use the substitution $u = \sin x$ and turn any unresolved powers of $\cos x$ into powers of $\sin x$ by using the following rearrangement of the Pythagorean identity: $\cos^2 x = 1 - \sin^2 x$, whose right-hand side is equal; to the expression $1 - u^2$ under the recommended substitution.
 - In cases where n is odd, we can use the substitution $u = \cos x$ and turn any unresolved powers of $\sin x$ into powers of $\cos x$ using this version of the Pythagorean identity: $\sin^2 x = 1 - \cos^2 x$, whose right-hand side is equal; to the expression $1 - u^2$ under the recommended substitution.
- If m and n are both even, we use the power-reduction formulae for $\sin^2 x$ and $\cos^2 x$ to rewrite the integrand as a series of powers of $\cos 2x$, which can then be integrated term by term, with the aid of the reduction formula for the integral of $\cos^n x$.

Note

The above examples have frequently made use of the close relationship between the trigonometric functions $\sin x$ and $\cos x$ – these functions are related by their behaviour when we differentiate them, as well as by the Pythagorean identity:

$$\sin^2 x + \cos^2 x = 1$$

which we have used above in various rearranged forms.

Elsewhere in this book we have remarked on the connections which exist between the trigonometric functions $\tan x$ and $\sec x$. They are connected in two distinct ways: first, by means of the trigonometric identity:

$$\sec^2 x - \tan^2 x = 1$$

along with its various rearrangements, and second by their relationship under differentiation:

$$\frac{d}{dx} \tan x = \sec^2 x \text{ and } \frac{d}{dx} \sec x = \sec x \tan x$$

We can use these relationships to integrate products of powers of $\tan x$ and $\sec x$, and we will now look at some examples of such integrals.

Example 8

We will start with the following integral:

$$I = \int \tan^3 x \sec^2 x dx$$

We can immediately recognise a function-derivative pair in the above integrand, so that it should be possible to integrate this example by substitution.

We will substitute as follows:

$$u = \tan x \text{ so that } \frac{du}{dx} = \sec^2 x \text{ and } dx = \frac{1}{\sec^2 x} du$$

Our integral is now transformed as follows:

$$\int u^3 \sec^2 x \frac{1}{\sec^2 x} du = \int u^3 du$$

which integrates easily to give:

$$\frac{1}{4}u^4 + C$$

Reversing the substitution produces our final solution in terms of x :

$$I = \frac{1}{4}\tan^4 x + C$$

Example 9

With that relatively simple example under our belts, let's look at something a stage more difficult:

$$I = \int \tan^3 x \sec^4 x dx$$

We will start by trying the same substitution as we used in [Example 8](#) – this was:

$$u = \tan x \text{ so that } \frac{du}{dx} = \sec^2 x \text{ and } dx = \frac{1}{\sec^2 x} du$$

This transforms our integral as follows:

$$\int u^3 \sec^4 x \frac{1}{\sec^2 x} du = \int u^3 \sec^2 x du$$

We still have an unresolved factor of $\sec^2 x$ in the new integrand. Fortunately, we can use a rearrangement of the identity linking $\tan x$ and $\sec x$ - the following rearrangement:

$$\sec^2 x = \tan^2 x + 1$$

to help us rewrite the transformed integral as follows:

$$\int u^3(\tan^2 x + 1)du = \int u^3(u^2 + 1)du = \int (u^5 + u^3)du$$

This integrates easily term by term as follows:

$$\frac{1}{6}u^6 + \frac{1}{4}u^4 + C$$

and on reversing the substitution we have the following final solution in terms of x :

$$I = \frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x + C$$

Note

The integrands in the preceding two examples have contained a factor of $\sec x$ raised to an even power, so that we were able to use the fact that the derivative of $\tan x$ is $\sec^2 x$ to facilitate the substitution $u = \tan x$, leaving us with even powers of $\sec x$ which we could convert to expressions in $\tan x$ using the trigonometric identity which connects these two functions. We will now look at the different approach which is called for when we have only an odd power of $\sec x$ in our integrand.

Example 10

The case we will deal with here is the following integral:

$$I = \int \tan^5 x \sec^3 x dx$$

In this case, we will adopt the substitution:

$$u = \sec x \text{ so that } \frac{du}{dx} = \sec x \tan x \text{ and so } dx = \frac{1}{\sec x \tan x} du$$

Our integral then transforms as follows:

$$\int \tan^5 x \sec^3 x \frac{1}{\sec x \tan x} du = \int \tan^4 x \sec^2 x du = \int u^2 \tan^4 x du$$

We can turn the outstanding factor in $\tan x$ into a more useful form by using the trigonometric formula linking the two functions currently under consideration, as follows:

$$\tan^4 x = (\tan^2 x)^2 = (\sec^2 x - 1)^2 = \sec^4 x - 2\sec^2 x + 1$$

With our chosen substitution, the final right-hand side above becomes:

$$u^4 - 2u^2 + 1$$

which allows us to finalise our integral in terms of u as follows:

$$\int u^2 (u^4 - 2u^2 + 1) du = \int (u^6 - 2u^4 + u^2) du$$

This integrates easily term by term to give us this solution in terms of u :

$$\frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C$$

and we can reverse out substitution to get the following final solution in terms of x :

$$\frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C$$

Note

Each of the previous three examples has featured an integrand containing an odd power of $\tan x$. To complete our survey of integrals of products of powers of $\tan x$ and $\sec x$, we will need to look at an example where an even power of $\tan x$ is present – this may prevent us from using some of the algebraic methods we have applied in conjunction with the expression we obtained for dx .

Example 11

The integral we will consider is the following:

$$I = \int \tan^2 x \sec^3 x dx$$

Here we have an even power of $\tan x$ in our integrand. A first step which might suggest itself would be to convert the square of $\tan x$ into an expression in $\sec x$ using the trigonometric identity which connects the two functions – we will then be left with powers of $\sec x$ alone. We will therefore apply the identity:

$$\tan^2 x = \sec^2 x - 1$$

to restate our integrand as follows:

$$(\sec^2 x - 1)\sec^3 x = \sec^5 x - \sec^3 x$$

Our integral is now reduced to the integral of a difference of two powers of $\sec x$, as follows:

$$\int (\sec^5 x - \sec^3 x) dx = \int \sec^5 x dx - \int \sec^3 x dx$$

This is actually a very useful outcome. We have integrated $\sec^3 x$ using integration by parts in [Part 2 Section 12](#), and we established a reduction formula which can be used to obtain the integral of any power of $\sec x$ in [Appendix 5 Section 11](#). Furthermore, this reduction formula is applied in [Appendix 4 Section 2](#) to create a table of the integrals of the first six powers of $\sec x$.

It's probably reasonable simply to quote the necessary results from the list in [Appendix 4 Section 2](#). For the component integrals we are dealing with here, these are as follows:

$$\int \sec^5 x dx = \frac{1}{4} \sec^3 x \tan x - \frac{3}{8} \sec x \tan x + \frac{3}{8} \ln |\sec x + \tan x| + C$$

$$\int \sec^3 x dx = \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln |\sec x + \tan x| + C$$

We can now complete our job by subtracting the second of these from the first. This gives us the following final solution:

$$I = \frac{1}{4} \sec^3 x \tan x - \frac{7}{8} \sec x \tan x + \frac{7}{8} \ln |\sec x + \tan x| + C$$

Summary for Powers of $\tan x$ and $\sec x$

We ought not to conclude without giving a brief summary of the different techniques which have been demonstrated for dealing with integrands comprising a product of powers of $\tan x$ and $\sec x$.

The integrals we have been considering in the previous three examples have been of the general form:

$$\int \tan^m x \sec^n x dx$$

and we have dealt with various different kinds of value for the variables m and n in three different ways. We can summarise these approaches and their applicability as follows:

- If n is even, as was the case in [Examples 8 and 9](#) above, we integrate using the substitution $u = \tan x$, which gives us the equivalence $du = \sec^2 x dx$; any remaining unresolved powers of $\sec x$ greater than 2 can then be converted into expressions in $\tan x$ using the identity $\sec^2 x = \tan^2 x + 1$.
- If n is odd, then we consider the two possible parities for the placeholder m :
 - If m is even, as we saw in [Example 11](#), we reduce the integrand to an expression in powers of $\sec x$ alone using the identity $\tan^2 x = \sec^2 x - 1$ and then appeal to the reduction formula for the integrals of powers of $\sec x$ to allow us to integrate the multiplied out expression term by term.
 - If m is odd, as in [Example 10](#), we integrate by substituting $u = \sec x$, from which we get the equivalence $du = \sec x \tan x dx$; any remaining unresolved powers of $\tan x$ can then be converted into expressions in $\sec x$ using the identity $\tan^2 x = \sec^2 x - 1$

Example 12: A Postscript

As a footnote to our look at these trigonometric integral, it is worth noting that our new-found ability to integrate some quite complicated combinations of trigonometric functions means that the technique of integration by parts which was dealt with in [Part 2](#) of this book can now be applied to a wider range of integrands. In order to see how this can be, we will look at just one example – a rather simple one at that, but chosen because it is an integral which is encountered in various branches of applied mathematics.

The integral we will consider is:

$$I = \int x \sin^2 x dx$$

Clearly, the trigonometric function which is here multiplied by a factor of x is by no means as complicated as some of the combinations we have been considering in this section, but it will serve as an illustration of the point I want to make without overcomplicating our working.

The integrand we are looking at is the product of two functions of different kinds, so it might be expected to be a good candidate for integration by parts. We will use the **LIATE** rule to choose parts as follows:

$$u = x \text{ and } \frac{dv}{dx} = \sin^2 x$$

From this, we can easily compute du/dx , but it is not so easy to see what the function represented by v should be. However, since we know dv/dx the Fundamental Theorem of Calculus, which tells us that integration allows us in a sense to reverse differentiation, should help us to see that:

$$v = \int \sin^2 x dx$$

This integral fits neatly into the context of this section since we can solve it with the aid of trigonometric manipulation – here, we will need to use the power-reduction formula:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

This lets us rewrite and easily evaluate the integral we have set out, in the following way:

$$v = \frac{1}{2} \int (1 - \cos 2x) dx = \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) + C = \frac{1}{2} x - \frac{1}{4} \sin 2x + C$$

We are now, finally, in a position to set down the two usual; consequences of our choice of parts. These are:

$$\frac{du}{dx} = 1 \text{ and } v = \frac{1}{2} x - \frac{1}{4} \sin 2x$$

With the aid of the trigonometric identity we have used, the integration by parts formula now gives us this equivalence:

$$\int x \sin^2 x dx = \frac{1}{2}x^2 - \frac{1}{4}x \sin 2x - \int \left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right) dx$$

The integral on the right-hand side integrates easily term by term to give the following solution:

$$\frac{1}{4}x^2 + \frac{1}{8}\cos 2x + C$$

so that the entire right-hand side of the integration by parts formula now looks like this:

$$\frac{1}{2}x^2 - \frac{1}{4}x \sin 2x - \frac{1}{4}x^2 - \frac{1}{8}\cos 2x + C$$

We can finish off by gathering together the terms in x^2 to get the following final solution to our original integral:

$$I = \frac{1}{4}x^2 - \frac{1}{4}x \sin 2x - \frac{1}{8}\cos 2x + C$$

Note

We should not fail to get the message from this small-scale example that any integral of the form:

$$\int x f(x) dx$$

where $f(x)$ stands for any of the combination trigonometric functions we have been integrating in this section, can now be integrated by parts – since we know how to integrate the function represented by $f(x)$. Integrating this function will be necessary, since the **LIATE** rule will make us choose :

$$u = x \text{ and } \frac{dv}{dx} = f(x) \text{ so that } v = \int f(x) dx$$

In fact, we can go a bit further and claim that, because of the relative simplicity of the structure of the product $x f(x)$, the solution to the general integral I have described will be as follows:

$$\int x f(x) dx = x F(x) - \int F(x) dx \text{ where } F(x) = \int f(x) dx$$

This is an immediate consequence of substituting the choice of parts I set out a moment ago into the right-hand side of the integration by parts formula, which we will recall is:

$$uv - \int \frac{du}{dx} v dx$$

We can immediately apply this formulaic rule to integrate a function which contains the trigonometric expression which we integrated in [Example 3](#) above in order to demonstrate this rule in action. If we start with the following integral:

$$I = \int x \sin^2 x \cos^3 x dx$$

and recall that the solution to [Example 3](#) told us that, using the notation of my general formula above:

$$f(x) = \sin^2 x \cos^3 x \text{ so that } F(x) = \int \sin^2 x \cos^3 x dx = \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x$$

(omitting for the moment the constant of integration) then all we have to do is to integrate the right-hand side immediately above in order to be able to write down the solution to our integration by parts problem.

Now this is, of course, more easily said than done: the expression we have to integrate contains a series of powers of $\sin x$: we could, of course, use the reduction formula for the integral of $\sin^n x$ to work out the required solution for each term in the series, or we could simply refer to the table of integrals of powers of $\sin x$ in [Appendix 4 Section 2](#). I intend to take the latter course in this instance.

We have to evaluate the following integral:

$$I_1 = \int F(x) dx = \int \left(\frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x \right) dx$$

Integrating term by term with the aid of our convenient table, we can set down the two terms of the solution as follows:

$$\begin{aligned} & \frac{1}{3} \left(-\frac{2}{3} \cos x - \frac{1}{3} \sin^2 x \cos x \right) \\ & -\frac{1}{5} \left(-\frac{8}{15} \cos x - \frac{4}{15} \sin^2 x \cos x - \frac{1}{5} \sin^4 x \cos x \right) \end{aligned}$$

These two terms multiply out as follows:

$$\begin{aligned} & -\frac{2}{9} \cos x - \frac{1}{9} \sin^2 x \cos x \\ & \frac{8}{75} \cos x + \frac{4}{75} \sin^2 x \cos x + \frac{1}{25} \sin^4 x \cos x \end{aligned}$$

We can now gather together like terms, raking care with our fractional arithmetic, and obtain the following solution to our intermediate integral:

$$I_1 = -\frac{26}{225}\cos x - \frac{13}{225}\sin^2 x \cos x + \frac{1}{25}\sin^4 x \cos x + C$$

Let's take a moment to gather together the facts we now have available which will let us write down our solution. My general formula predicted the following form for a solution to the integral I :

$$I = \int xf(x)dx = xF(x) - \int F(x)dx \text{ where } F(x) = \int f(x)dx$$

We have now established the following (and I will temporarily omit the constant of integration here, only to add it back again as required at the end of our working):

$$F(x) = \int f(x)dx = \int \sin^2 x \cos^3 x dx = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x$$

so that:

$$xF(x) = \frac{1}{3}x\sin^3 x - \frac{1}{5}x\sin^5 x$$

Furthermore, we now know that:

$$\int F(x)dx = I_1 = -\frac{26}{225}\cos x - \frac{13}{225}\sin^2 x \cos x + \frac{1}{25}\sin^4 x \cos x$$

The solution we want is therefore:

$$I = \frac{1}{3}x\sin^3 x - \frac{1}{5}x\sin^5 x + \frac{26}{225}\cos x + \frac{13}{225}\sin^2 x \cos x - \frac{1}{25}\sin^4 x \cos x + C$$

Clearly, using my “standard” formula in the current context cannot be done without some labour; it does, however, perhaps serve to keep that labour to a minimum.

We might conclude with two observations: first, the above general rule based on the integration by parts technique can be applied more widely than just to situations where the function $f(x)$ is a trigonometric function – it also applies to cases where $f(x)$ can be classified as an exponential function, since the **LIATE** rule recommends us to choose parts in these cases in precisely the same order as with our current trigonometric instances of $f(x)$.

Second – and this may be considered to constitute a limitation to the rule's use – if we look carefully at the right-hand side of the rule of integration, which we will recall is:

$$xF(x) - \int F(x)dx \text{ where } F(x) = \int f(x)dx$$

then we will note that in order to use it, we must be able to integrate $F(x)$, which is of course the integral of $f(x)$, the function which appears in the original integral. This means that to be

able to use the rule, we will need to be sure that we can integrate the function $f(x)$ not just once, but twice!

Postscript

How to Integrate Almost Anything

Having arrived at the end of the book, the time is now ripe for a few comments to put the various techniques we have examined into some sort of perspective.

In [Section 4](#) of the introduction, I mentioned the various techniques, or “rules” which we are taught to use in order to differentiate functions which appear as various combinations of the simpler functions we can differentiate straight away.

We ought to be able to see, in the light of this book’s contents, that we actually have a rather larger armoury of techniques for use in integrating functions which won’t respond to the simple rules. These techniques are, however, much more “special-purpose” than their counterparts from the world of differentiation – in other words, their field of application is much more circumscribed.

For example, we do not have a simple integration rule to deal with products of functions – we do, however, have simple u -substitution, which we can use when a product of functions contains a function-derivative pair in the correct configuration; and we have integration by parts, which deals with products of functions of different types under the **LIATE** classification explained in [Part 2 Section 7](#). [Appendix 2](#) contains a fairly comprehensive table which shows the kinds of integrand which will respond to each of the various techniques explored in the text.

It is natural to ask: is there a simple heuristic which can be used to decide how to approach the integration of a function which we are encountering for the first time? I will attempt here to offer some thoughts on how such a heuristic might be constructed.

First of all, it is important to realise at the outset that some integrands can be dealt with by one single application of a particular technique, while others will require an iterative approach – in such cases, the application of one technique leads to an integrand where the same or another technique may be needed subsequently in order to get closer to a final solution.

Having made this point, let’s see if we can set out a broad “plan of action” for dealing with any new integration problem where our simple rules are not powerful enough to do the trick.

Try This First

The first step should always be to try applying u -substitution. This technique has quite wide applicability. We might first ask whether the simplest kind of substitution – of the form $u = ax + b$ – can be used. If not, we should look to see whether a function-derivative pair is present – this means asking whether the integrand has the form $kg'(f(x))f'(x)$.

Try This Next

Our next step, if the integrand we are faced with fails the tests implied in the previous paragraph, ought to be to ask if it is the product of two different types of function from the **LIATE** list – this list and its importance are described in detail in [Part 2 Section 7](#). If the integrand fits this model, then we should expect to apply integration by parts successfully. It's worth bearing in mind that integration by parts may have to be applied repeatedly (this can be done neatly using the tabular approach described in [Part 2 Section 6](#)). Some combinations of functions – after repeated integration by parts - will reappear on the right-hand side of our working. This will frequently yield an equation which can be solved to produce a final integral.

Does a Special-Purpose Method Apply?

If neither of the approaches described above can be applied, we must see whether or not a special-purpose approach can be applied. If our integrand contains square or cube roots of various kinds, then we can apply trigonometric substitution (discussed in [Part 1 Section 10](#)), and the useful technique of power-substitution (discussed in [Part 1 Section 9](#) and again in [Part 3](#) as part of a multi-stage process).

Another kind of special-purpose technique is the use of integration by parts to integrate inverse trigonometric or logarithmic functions by treating them as products with the very simple “function” 1, and then setting dv/dx equal to 1. This method is described and examples given in [Part 2 Section 8](#).

Do We Need to Make an Algebraic Rearrangement?

If none of our special-purpose techniques will meet the need, we may have to consider some algebraic rearrangement of the integrand in order to rewrite the integrand in such a way as to bring it within the range of one of our techniques. This kind of preparation of the integrand (not generally something we have to do when differentiating, be it noted) includes the use of such techniques as completing the square (used in [Part 1 Section 12](#)) as well as the simple algebraic manipulation I have called “converting rational expressions” (used several times in [Part 1](#), and explained in [Appendix 3 Section 2](#)).

If we have to integrate a rational function whose numerator and denominator are both polynomials, then we have at our disposal the Method of Partial Fractions and polynomial division – both examined in great detail in [Part 3](#). The correct one of these two to use will be dictated by the degrees of the polynomials in the numerator and denominator as well as whether the denominator polynomial can be factorised – [Appendix 1 Sections 3 and 4](#) provide algorithms which show the domain of applicability of each technique.

Do We Need to Make a Trigonometric Rearrangement?

If we have to integrate a product of trigonometric functions, we saw in [Part 3 Section 9](#) that the rich set of trigonometric identities can help us prepare such integrands for integration. A number of methods are available, depending on the precise nature of the integrand we are dealing with.

Don't Forget That More Than One Technique May be Needed

It is so important to realise that the integration of some functions has to be an iterative process, where one technique after another, or even the same technique repeatedly, must be applied – so important that this is the second time I have made this point in the course of this

postscript.

Appendix 1

Algorithms

1 Integration by Substitution

I summarise below the algorithm for basic u -substitution which is given along with detailed explanatory notes in [Part 1 Section 4](#).

Step 1: Choose the substitution u in terms of x .

Try to spot a “function-derivative pair”. It can sometimes be more convenient or informative to express x in terms of u .

Step 2: Obtain an expression for dx .

We do this by first considering either du/dx or dx/du , depending on how our substitution formula is expressed.

$$\text{If } \frac{du}{dx} = f(x) \text{ then } dx = \frac{du}{f(x)}$$

$$\text{If } \frac{dx}{du} = f(u) \text{ then } dx = f(u)du$$

Step 3: Make the substitution both for the integrand and for dx .

In order to do this, it may be necessary to rearrange our substitution formula, which links u and x , so that the subject of the formula is changed to the other of these two variables.

Step 4: Integrate the resulting integral with respect to u ,

Step 5: Reverse the substitution to restate the solution in terms the original variable x .

2 Integration by Parts

Here is a summary of the algorithms for integration by parts which are set out with detailed explanation in [Part 2 Section 9](#).

The “Basic” Technique

Step 1: Choose parts u and dv/dx .

The integrand must take the form:

$$\int u \frac{dv}{dx} dx$$

Choose parts using the **LIATE** mnemonic – the letters represent the following types of function:

Letter	Meaning	Examples
L	Logarithmic	$\ln x, \log_a x$

I	Inverse trigonometric	$\sin^{-1} x, \tan^{-1} x$
A	Algebraic	$x^2, 6x^4$
T	Trigonometric	$\cos x, \tan x$
E	Exponential	$e^x, 2^x$

Step 2: Evaluate du/dx and v .

Step 3: Apply the formula.

The integration by parts formula is as follows:

$$\int u \frac{dv}{dx} dx = uv - \int \frac{du}{dx} v dx$$

Step 4: Evaluate the resulting right-hand side integral.

Four possible cases will arise:

Case 1: The integral can be integrated simply.

Case 2: The integral requires a specific technique.

Case 3: The integral must be integrated by parts.

Case 4: The original integral reappears.

The Tabular Method

Step 1: Choose parts u and dv/dx .

As in the “basic” technique, the integrand must take the form:

$$\int u \frac{dv}{dx} dx$$

We again choose parts using the **LIATE** mnemonic, set out above, and with this technique we expect that the function we choose as u will be reducible by repeated differentiation to zero.

Step 2: Create the blank table.

We create our blank table with three columns, headed as follows:

u	$v (dv/dx = f(x))$	$Sign$

Here, $f(x)$ represents the part of our integrand we have chosen to be dv/dx .

Step 3: Complete the table.

In the first row, the entry for u will come from our choice of parts; the entry for v will be obtained from the value for dv/dx in the table heading; and the sign will be a minus sign.

We subsequently complete each row as follows: the u entry comes from differentiating the u entry in the preceding row; the v entry comes from integrating the v entry in the preceding row; and the sign entry comes from reversing the sign in the previous row.

The process ends when the u entry is found to be zero.

Step 4: Read off the solution.

We read off the solution by writing down the product uv followed by the entry in the *Sign* column for each line in turn. This continues until the presence of zero in the u column marks the end of the tabular process.

3 The Method of Partial Fractions

This is an algebraic technique which can be used to transform some rational expressions into an expanded form which can be integrated more easily than the original expression.

Step 1: Determine the type of the rational expression

Rational expressions of the following types are treated in [Part 3](#) of this book – beside each type is shown the kind of expansion the technique will give us:

Type of Rational Expression	Pattern of the Expansion
$\frac{px + q}{(x + a)(x + b)}$	$\frac{A}{x + a} + \frac{B}{x + b}$
$\frac{px + q}{(x + a)(x + b)(x + c)}$	$\frac{A}{x + a} + \frac{B}{x + b} + \frac{C}{x + c}$
$\frac{px^2 + qx + r}{(x + a)(x + b)(x + c)}$	$\frac{A}{x + a} + \frac{B}{x + b} + \frac{C}{x + c}$
$\frac{ps + q}{(x + a)(x + b)^2}$	$\frac{A}{x + a} + \frac{Bx + C}{(x + b)^2}$ or $\frac{A}{x + a} + \frac{B}{x + b} + \frac{C}{(x + b)^2}$
$\frac{px^2 + qx + r}{(x + a)(x^2 + bx + c)}$	$\frac{A}{x + a} + \frac{Bx + C}{x^2 + bx + c}$
$\frac{px + q}{(x + a)(x^2 + bx + c)^2}$	$\frac{A}{x + a} + \frac{Bx + C}{x^2 + bx + c} + \frac{Dx + E}{(x^2 + bx + c)^2}$

Step 2: Form a numerator from the expanded version

Starting with the appropriate expanded version containing the placeholders A, B etc, add the separate fractional terms together using a common denominator.

Step 3: Equate numerators

Equate the numerator of the expression obtained at Step 2 with the numerator of the original rational expression.

Step 4: Determine the values of the placeholders

Two methods for completing this step are explained in the text.

Method 1: Choose suitable values for x in the equation of numerators so that all the placeholders except one are multiplied by zero; then solve the equation in the remaining placeholder to obtain its value. (*Note: this may not be possible for rational expressions of the third, fourth or fifth types in the above table.*)

Method 2: Multiply out the numerator obtained at Step 2 and gather together like terms. Equate coefficients for each power of x present on either side of the equation. Solve the resulting system of equations to obtain values for each of the placeholders.

(*Note: a third method, known as the “cover-up method”, is discussed in [Appendix 3](#).*)

4 The Tabular Method of Polynomial Division

The text contains explanations of a tabular approach to polynomial division for two particular types of divisor: linear divisors and quadratic divisors.

Polynomial Division by a Linear Divisor

This class of polynomial division can be expressed using the following formula:

$$\frac{P(x)}{ax + b} = Q(x) + \frac{r}{ax + b}$$

The numerator polynomial on the left-hand side has the following form:

$$P(x) = p_n x^n + p_{n-1} x^{n-1} + \dots + p_2 x^2 + p_1 x + p_0$$

and the quotient polynomial on the right-hand side has the following form:

$$Q(x) = q_{n-1} x^{n-1} + q_{n-2} x^{n-2} + \dots + q_2 x^2 + q_1 x + q_0$$

The tabular method of carrying out the division operation allows us to derive the values of the quotient coefficients q_i and the remainder constant r from the numerator polynomial's coefficients p_i . This is done by solving a series of simple equations.

Step 1: Write Down the Coefficients of the Denominator Expression

Write down for reference the values of the constants a and b .

Step 2: Create the Blank Table

The format for the blank table is as follows:

<i>Power of x</i>	<i>Coefficient of $P(x)$</i>	<i>Equation of Coefficients</i>	<i>Solution</i>

Step 3: Complete Column 1

Make an entry in column 1 for every power of x beginning with the highest power in the numerator polynomial, x^n and continuing all the way down to x^2 , x and finally 1 (or x^0). Every power of x from the highest power n down to 0 must have an entry, even if some of them do not appear explicitly in the polynomial $P(x)$.

Step 4: Complete Column 2

Complete column 2 by making an entry in each line of the form $p_i = k$, where p_i is the coefficient of the term in x^i in the polynomial $P(x)$. Each coefficient should appear on the same line as its corresponding power of x . For a power of x which does not explicitly appear, enter a coefficient of 0.

Step 5: Complete Column 3

Using the following reference table:

Power of x	Equation of Coefficients
x^n	$aq_{n-1} = p_n$
x^i where $0 < i < n$	$aq_{i-1} + bq_i = p_i$
1	$bq_0 + r = p_0$

enter on each line the appropriate equation of coefficients, substituting any values which have already been written down or obtained from an earlier equation. The result will be a series of simple linear equations, each one in a single variable.

Step 6: Complete Column 4

For each of the equations in column 3, enter its solution in the corresponding line of column 4. This will result in a column containing explicit values for each of the coefficients q_i and the remainder constant r .

Step 7: Write Down the Result

The result of the polynomial division can now be written down as follows:

$$q_{n-1}x^{n-1} + q_{n-2}x^{n-2} \dots + q_2x^2 + q_1x + q_0 + \frac{r}{ax + b}$$

substituting in all the values for the coefficients q_i and r from column 4 of the completed table.

Polynomial Division by a Quadratic Divisor

This class of polynomial division can be expressed using the following formula:

$$\frac{P(x)}{ax^2 + bx + c} = Q(x) + \frac{R(x)}{ax^2 + bx + c}$$

The numerator polynomial on the left-hand side has the following form:

$$P(x) = p_nx^n + p_{n-1}x^{n-1} + \dots + p_2x^2 + p_1x + p_0$$

and the quotient polynomial on the right-hand side has the following form:

$$Q(x) = q_{n-2}x^{n-2} + q_{n-3}x^{n-3} \dots + q_2x^2 + q_1x + q_0$$

The numerator of the remainder term is a linear expression in x , which has the following form:

$$R(x) = r_1x + r_0$$

The tabular method of carrying out the division operation allows us to derive the values of the quotient coefficients q_i and the remainder coefficients r_i from the numerator polynomial's coefficients p_i . This is done by solving a series of simple equations.

Step 1: Write Down the Coefficients of the Denominator Expression

Write down for reference the values of the constants a , b and c .

Step 2: Create the Blank Table

The format for the blank table is as follows:

<i>Power of x</i>	<i>Coefficient of $P(x)$</i>	<i>Equation of Coefficients</i>	<i>Solution</i>

Step 3: Complete Column 1

Make an entry in column 1 for every power of x beginning with the highest power in the numerator polynomial, x^n and continuing all the way down to x^2 , x and finally 1 (or x^0). Every power of x from the highest power n down to 0 must have an entry, even if some of them do not appear explicitly in the polynomial $P(x)$.

Step 4: Complete Column 2

Complete column 2 by making an entry in each line of the form $p_i = k$, where p_i is the coefficient of the term in x^i in the polynomial $P(x)$. Each coefficient should appear on the same line as its corresponding power of x . For a power of x which does not explicitly appear, enter a coefficient of 0.

Step 5: Complete Column 3

Using the following reference table:

<i>Power of x</i>	<i>Equation of Coefficients</i>
x^n	$aq_{n-2} = p_n$
x^{n-1}	$aq_{n-3} + bq_{n-2} = p_{n-1}$
x^i where $1 < i < n - 1$	$aq_{i-2} + bq_{i-1} + cq_i = p_i$
x	$bq_0 + cq_1 + r_1 = p_1$
1	$cq_0 + r_0 = p_0$

enter on each line the appropriate equation of coefficients, substituting any values which have already been written down or obtained from an earlier equation. The result will be a series of simple linear equations, each one in a single variable.

Step 6: Complete Column 4

For each of the equations in column 3, enter its solution in the corresponding line of column 4. This will result in a column containing explicit values for each of the coefficients q_i and the remainder coefficients r_i .

Step 7: Write Down the Result

The result of the polynomial division can now be written down as follows:

$$q_{n-2}x^{n-2} + q_{n-3}x^{n-3} \dots + q_2x^2 + q_1x + q_0 + \frac{r_1x + r_0}{ax^2 + bx + c}$$

substituting in all the values for the coefficients q_i and r_i from column 4 of the completed table.

It will be useful to have to hand some explicit polynomial division tables which we can use without the need to interpret the general line in terms of the subscript i which appears in both

the reference tables I have included in the above algorithms. Some examples of these are listed below.

Reference Tables for Specific Numerators with a Linear Denominator

1 Quadratic Numerator Divided by Linear Denominator

To carry out polynomial divisions of the following form:

$$\frac{p_2x^2 + p_1x + p_0}{ax + b} = q_1x + q_0 + \frac{r}{ax + b}$$

the table below can be used:

Power of x	Equation of Coefficients
x^2	$aq_1 = p_2$
x	$aq_0 + bq_1 = p_1$
1	$bq_0 + r = p_0$

2 Cubic Numerator Divided by Linear Denominator

To carry out polynomial divisions of the following form:

$$\frac{p_3x^3 + p_2x^2 + p_1x + p_0}{ax + b} = q_2x^2 + q_1x + q_0 + \frac{r}{ax + b}$$

the table below can be used:

Power of x	Equation of Coefficients
x^3	$aq_1 = p_3$
x^2	$aq_1 + bq_2 = p_2$
x	$bq_0 + cq_1 = p_1$
1	$bq_0 + r = p_0$

Reference Tables for Specific Numerators with a Quadratic Denominator

1 Cubic Numerator Divided by Quadratic Denominator

To carry out polynomial divisions of the following form:

$$\frac{p_3x^3 + p_2x^2 + p_1x + p_0}{ax^2 + bx + c} = q_1x + q_0 + \frac{r_1x + r_0}{ax^2 + bx + c}$$

the table below can be used:

Power of x	Equation of Coefficients
x^3	$aq_1 = p_3$
x^2	$aq_0 + bq_1 = p_2$
x	$bq_0 + cq_1 + r_1 = p_1$
1	$cq_0 + r_0 = p_0$

2 Quartic Numerator Divided by Quadratic Denominator

To carry out polynomial divisions of the following form:

$$\frac{p_4x^4 + p_3x^3 + p_2x^2 + p_1x + p_0}{ax^2 + bx + c} = q_2x^2 + q_1x + q_0 + \frac{r_1x + r_0}{ax^2 + bx + c}$$

the table below can be used:

<i>Power of x</i>	<i>Equation of Coefficients</i>
x^4	$aq_2 = p_4$
x^3	$aq_1 + bq_2 = p_3$
x^2	$aq_0 + bq_1 + cq_2 = p_2$
x	$bq_0 + cq_1 + r_1 = p_1$
1	$cq_0 + r_0 = p_0$

Appendix 2

When to Use Each Technique

1 Integration by Substitution

The following table sets out the various types of integrand to which we have applied the technique of substitution in its various forms. Against each type of integrand are some brief comments on how best to apply substitution.

<i>Integral</i>	<i>Recommended Method</i>
$\int kg'(f(x))f'(x)dx$	This is the standard form of integral which can be solved by straightforward <i>u</i> -substitution, setting $u = f(x)$.
$\int F(ax + b)dx$	This type constitutes a special case of the form given above, where $g'(f(x)) = F(ax + b)$, $f(x) = ax + b$, $f'(x) = a$, $k = 1/a$.
$\int (ax + b)(cx + d)^p dx$	This form of integral can be solved using the substitution $u = cx + d$.
$\int \frac{ax^{2n-1}}{bx^n + c} dx$	Integrals of this form can be solved using the substitution $u = bx^n + c$ or alternatively $u = bx^n$
$\int g(\sqrt[n]{f(x)})dx$	Integrals of this form respond to a power substitution of the form $f(x) = u^n$
$\int f(\sqrt{a^2 - x^2})dx$	Integrals of this form can be solved using sine-substitution , typically $x = a \sin u$.
$\int f(\sqrt{x^2 + a^2})dx$	Integrals of this form can be solved using tangent-substitution , typically $x = a \tan u$.
$\int f(\sqrt{x^2 - a^2})dx$	Integrals of this form can be solved using secant-substitution , typically $x = a \sec u$.
$\int \frac{Ax + B}{\sqrt{Cx^2 + Dx + E}} dx$ $\int \frac{Ax + B}{Cx^2 + Dx + E} dx$ $\int \sqrt{Ax^2 + Bx + C} dx$	These are solved by first completing the square for the quadratic expression, followed by repeated substitution. The second substitution may be trigonometric or algebraic. Where A in the numerator is non-zero, it may be necessary after the first substitution to split the integrand into two fractions and integrate each separately with a different substitution.

2 Integration by Parts

Similarly, the following table summarises our experience with integration by parts – each class of function is accompanied by some notes for guidance. In this table, $P(x)$ represents some polynomial in x . In all cases, we should choose parts using the **LIATE** rule.

<i>Integral</i>	<i>Recommended Method</i>
$\int xf(x)dx$	One application of integration by parts will be expected to work if $f(x)$ is exponential, trigonometric, inverse trigonometric or logarithmic.
$\int P(x)f(x)dx$	Repeated integration by parts will be needed if $f(x)$ is trigonometric or exponential.
$\int P(x)f(x)dx$	One application of integration by parts should work if $f(x)$ is inverse trigonometric or logarithmic.
$\int 1 \cdot f(x)dx$	Treating the integrand as shown and integrating by parts is indicated if $f(x)$ is inverse trigonometric or logarithmic.

3 Partial Fractions

In the following table we have a summary of the types of integrand which can be integrated fairly easily once they have been transformed into an expanded, multi-term form using the algebraic Method of Partial Fractions. The recommended expansion for each type of integrand is given, together with some brief notes on how the process of integration can be completed on the transformed integrand.

<i>Integral</i>	<i>Recommended Expansion and Integration Method</i>
$\int \frac{px + q}{(x + a)(x + b)} dx$	$\frac{A}{x + a} + \frac{B}{x + b}$ Integrate term by term using appropriate integration rule derived from the substitution technique.
$\int \frac{px + q}{(x + a)(x + b)(x + c)} dx$	$\frac{A}{x + a} + \frac{B}{x + b} + \frac{C}{x + c}$ Integrate term by term as for the first type above, applying the same rule of integration.
$\int \frac{px^2 + qx + r}{(x + a)(x + b)(x + c)} dx$	$\frac{A}{x + a} + \frac{B}{x + b} + \frac{C}{x + c}$ Integrate term by term as for the first and second types above, applying the same rule of integration
$\int \frac{ps + q}{(x + a)(x + b)^2} dx$	$\frac{A}{x + a} + \frac{Bx + C}{(x + b)^2} \text{ or } \frac{A}{x + a} + \frac{B}{x + b} + \frac{C}{(x + b)^2}$ Terms with constant numerators can be integrated as with the terms of the preceding three types. The final term in the first expansion is integrated by substitution. The final term in the second expansion is integrated using one of the rules of integration derived from substitution.
$\int \frac{px^2 + qx + r}{(x + a)(x^2 + bx + c)} dx$	$\frac{A}{x + a} + \frac{Bx + C}{x^2 + bx + c}$ The first term is integrated as with the terms in the first three types in the table. The second term is integrated by completing the square for the denominator and then using substitution.
$\int \frac{px + q}{(x + a)(x^2 + bx + c)^2} dx$	$\frac{A}{x + a} + \frac{Bx + C}{x^2 + bx + c} + \frac{Dx + E}{(x^2 + bx + c)^2}$ The first term is integrated as with the terms in the first three types in the table. The second term is integrated by completing the square for the denominator and then using substitution. The third term should be split into two parts and then evaluated by substitution (including completing the square for the denominator and using a trigonometric substitution where necessary).

4 Polynomial Division

In the text, we explained the integration of two classes of rational expression where polynomial division was used to transform the integrand into a form which could be dealt with more easily. The two classes of expression are listed below – in the left-hand column abbreviations (in order for polynomial division to be an appropriate technique), the degree of the numerator polynomial $P(x)$ is assumed to be greater than the degree of the denominator expression. In the right-hand column, the quotient expression $Q(x)$ is a polynomial of lower degree than $P(x)$ and on the second row, the remainder expression $R(x)$ is a linear expression in x .

<i>Integral</i>	<i>Recommended Expansion and Integration Method</i>
$\int \frac{P(x)}{ax + b} dx$	$Q(x) + \frac{r}{ax + b}$
	Integrate $Q(x)$ term by term and integrate the final term using one of the rules of integration derived from the method of substitution
$\int \frac{P(x)}{ax^2 + bx + c} dx$	$Q(x) + \frac{R(x)}{ax^2 + bx + c}$
	Integrate $Q(x)$ term by term. The final term should be split into two parts and then evaluated by substitution (including completing the square for the denominator and using a trigonometric substitution where necessary).

5 Trigonometric Integrals

The following table lists the kinds of trigonometric integral examined in [Part 3 Section 9](#) along with the recommended approach to their evaluation. We first list integrals of products of sine and cosine expressions whose arguments are simple multiples of x

<i>Integral</i>	<i>Recommended Method</i>
$\int \sin mx \cos nx \, dx$	Use the appropriate product-to-sum formula to convert the integrand to a sum of two sine functions – then integrate term by term
$\int \sin mx \sin nx \, dx$	Use the appropriate product-to-sum formula to convert the integrand to a difference of two cosine functions – then integrate term by term
$\int \cos mx \cos nx \, dx$	Use the appropriate product-to-sum formula to convert the integrand to a sum of two cosine functions – then integrate term by term

The next table lists integrals whose integrands are products of powers of trigonometric functions.

<i>Integral</i>	<i>Recommended Method</i>
$\int \sin^m x \cos^n x \, dx$	If m is odd
	Use the substitution $u = \sin x$ and transform unresolved cosines to sines using the Pythagorean identity
	If n is odd
	Use the substitution $u = \cos x$ and transform unresolved sines to cosines using the Pythagorean identity
	If m and n are both even
	Use the power-reduction formulae for $\sin^2 x$ and $\cos^2 x$ to rewrite the integrand as a series of powers of $\cos 2x$ and integrate term by term with the help of the reduction formula for $\cos x$
$\int \tan^m x \sec^n x \, dx$	If n is even
	Use the substitution $u = \tan x$ and transform unresolved secants to tangents using the identity which links these two functions
	If n is odd and m is even
	Use the identity which links $\tan x$ and $\sec x$ to rewrite the integrand as an expression in powers of $\sec x$ alone then integrate term by term using the reduction formula for $\sec x$
	If n is odd and m is odd
	Use the substitution $u = \sec x$ and transform unresolved tangents to secants using the identity which links these two functions

Appendix 3

Useful Algebraic Methods

1 Completing the Square

This is a commonly used technique for rearranging a quadratic expression into a new form which can be used more easily in a variety of contexts.

The general form of a quadratic expression in x is as follows:

$$ax^2 + bx + c$$

I propose to explain this technique as a two-part process. First, I will examine the case of a *monic* quadratic expression, ie one where, in the above general form, the coefficient a is equal to 1. I will then proceed to the more general non-monic case, where a takes a value other than 1.

Monic Quadratics

A monic quadratic expression in x has the form:

$$x^2 + bx + c$$

sticking with the naming of the coefficients adopted above. Using the process of completing the square, I propose to rearrange this expression into the form:

$$(x + p)^2 + q$$

We will demonstrate that this is possible, while at the same time providing a methodology for carrying out the process, by deriving expressions for the constants p and q in terms of b and c .

We will do this by expanding out our expression in p and q , and then equating the coefficients of the x and constant terms in the quadratic to b and c .

Our first step is to multiply out the squared expression in brackets. This converts our expression in p and q into:

$$x^2 + 2px + p^2 + q$$

giving us the following two equivalences:

$$\begin{aligned} b &= 2p \\ c &= p^2 + q \end{aligned}$$

Rearranging the first of these, we obtain the following expression for p :

$$p = \frac{b}{2}$$

Substituting this into the expression for c and rearranging in order to make q the subject, we obtain:

$$q = c - \frac{b^2}{4}$$

Making these substitutions for p and q in the expression into which we want to convert our quadratic, we now have the following formula for completing the square:

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)$$

Non-Monic Quadratics

A non-monic quadratic expression takes the form:

$$ax^2 + bx + c$$

where a has a value other than 1. (We further assume, incidentally, that a is non-zero, since otherwise we are not dealing with a quadratic expression at all!) We will aim to establish a way of converting quadratics of this type into the following form:

$$a(x + p)^2 + q$$

As before, we will proceed by expanding out our desired expression in p and q in order to obtain expressions for p and q , this time in terms of a , b and c .

We start by expanding out the squared term in brackets. This gives us:

$$a(x^2 + 2px + p) + q$$

We must now multiply the contents of the bracketed expression by a , to obtain the following:

$$ax^2 + 2apx + ap^2 + q$$

In this expression, the coefficient of x^2 is still a , just as it was in the standard form of the non-monic quadratic. In addition, we can now write down expressions for b and c in terms of the other variables. These expressions are as follows:

$$\begin{aligned} b &= 2ap \\ c &= ap^2 + q \end{aligned}$$

Rearranging the first of these, we obtain the following expression for p :

$$p = \frac{b}{2a}$$

Substituting this into the expression for c and rearranging in order to make q the subject, we obtain:

$$q = c - \frac{b^2}{4a}$$

Making these substitutions for p and q in the expression into which we want to convert our quadratic, we now have the following formula for completing the square:

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$$

Notes

An examination of the formulae we have derived, which can now be used to complete the square for any quadratic expression, will show that there are only some circumstances in which the constants in the “new” form of the quadratic will be whole numbers. In the case of the monic quadratic, this will be expected to happen when b is even. In the non-monic case, we require that b should not merely be even but indeed should be exactly divisible by $2a$.

The main reason for explaining this algebraic technique here is that it is useful in processing quadratic expressions in an integrand into a form where u -substitution can be used. This is particularly relevant in the cases discussed in [Part 1 Section 12](#).

However, it should be pointed out that the technique is also useful in drawing the graph of an arbitrary quadratic function and in finding the roots of quadratic equations, among other perhaps more advanced applications.

Postscript: Factorised Quadratics

As a final example of this technique, it will be interesting to derive a formula for completing the square when we are presented with a quadratic expression in factorised form, ie the following form:

$$(x + a)(x + b)$$

(I only intend to cover the monic case here, for the sake of simplicity.)

In order to derive our formula, we can of course take full advantage of the formula we obtained above for the monic quadratic expressed as a polynomial. In order to do this, we need first to expand the factorised expression I set out a few lines ago into polynomial form. Simple algebraic multiplying-out will give us this result:

$$x^2 + (a + b)x + ab$$

We now recall our formula for completing the square where we are given a monic quadratic as a polynomial. Using capitalized constants to avoid ambiguity in connection with our factorised quadratic expression and its expansion, the necessary formula is as follows:

$$x^2 + Bx + C = \left(x + \frac{B}{2}\right)^2 + \left(C - \frac{B^2}{4}\right)$$

Looking back at the expansion of the factorised quadratic we are considering here, and equating coefficients, we can state these two equalities:

$$B = a + b \text{ and } C = ab$$

Substituting these into the right-hand side of our formula for completing the square, we arrive at the following final formula for completing the square when we start with a factorised quadratic expression:

$$(x + a)(x + b) = \left(x + \frac{a + b}{2}\right)^2 + \left(ab - \frac{(a + b)^2}{4}\right)$$

2 Converting Rational Expressions

In [Part 2 Section 4 Example 4](#), we eventually had to evaluate the following integral:

$$\int \frac{x^2}{x^2 + 1} dx$$

We managed this by converting the rational expression in the integrand into something of the form:

$$1 \pm \frac{1}{(\text{some expression in } x)}$$

which allowed us to complete the job relatively easily.

What I want to do here is to explain the general principles of the algebraic manipulation we used to achieve this transformation, and express it in a more general form, which could prove useful in approaching a range of integrals.

The Basic Method

Let's start by considering an expression based in the example quoted above, but with slightly more generality. We will start with:

$$\frac{x^n}{x^n + a}$$

and see how we might turn it into the kind of mixed expression which helped us with our concrete example.

Actually, the process is relatively simple. Seeing that we want to change our rational expression into the form:

$$1 \pm \frac{(\text{some constant})}{(\text{some expression in } x)}$$

we might start by looking at what adjustment our expression would need in order to change it into something which is equivalent to 1, the first part of our target form.

Very simply, this can be achieved by making the numerator equal to the denominator – in our specific case, by adding a to the numerator. This would result in the following expression:

$$\frac{x^n + a}{x^n + a} = 1$$

This would be all very convenient if it were possible. However we must immediately object that arbitrarily adding a to the numerator will change the value of the expression without any justification.

In order to deal with this objection, however, I can offer the following approach: having added a to the numerator, we can ensure that the value of our expression remains the same by immediately subtracting a from the numerator. Adding and subtracting the same quantity to an expression leaves its value unaltered. We might quote a general rule of algebra to the following effect:

$$a + b - b = a$$

If we are prepared to make this simple change, we will find that our rational expression has now taken on the following form:

$$\frac{x^n + a - a}{x^n + a}$$

This may seem a pointless exercise, until we consider that we can now separate our expression into a sum of two rational expressions with the same denominator, each of which is easier to handle from the viewpoint of integration than the expression we started with. The separation we want (which should leave us with one rational expression which reduces by cancellation to 1) is as follows:

$$\frac{x^n + a}{x^n + a} - \frac{a}{x^n + a}$$

Cancelling the numerator and denominator in the first term above will bring us to the following form for our original rational expression:

$$1 - \frac{a}{x^n + a}$$

This approach constitutes a kind of “*borrowing and paying-back*” which is not uncommon in other areas of mathematics – one thinks, for example, of the simple trick of “*borrowing*” which is done on a routine basis in arithmetic when carrying out subtraction by hand.

We have transformed our numerator into an equivalent expression which leaves our way clear to a separation into two fractions, one of which cancels to 1, and the second of which has all terms in x removed from its numerator, giving us a slightly easier ride when we come to integrate the expression.

A Useful Generalisation

We can take this method one stage further by considering the following expression:

$$\frac{x^n + a}{x^n + b}$$

where $a \neq b$. (If these were equal, then we would be dealing with an expression which cancelled down to a value of 1 immediately!) Can we transform this expression in the same way we handled the simpler one in the subsection above?

In fact, the answer is yes. First, we ask how we would change the numerator in order to arrive at an expression which cancels down to a value of 1. To achieve this, we would have to add $b - a$ to the numerator, in order to remove the a term and substitute a b term. This would have the following result:

$$\frac{x^n + a + b - a}{x^n + b} = \frac{x^n + b}{x^n + b}$$

(It's easy to see that the two instances of a cancel each other out.)

So far, so good – but we are still left with the uncomfortable awareness that this has changed the value of our original expression. In order to allay these concerns, we can “pay back” the “borrowed” $b - a$ by *subtracting* it from the new numerator, just as we did with the simple term a in the previous case.

This means that our original rational expression can now be rewritten as:

$$\frac{x^n + a + (b - a) - (b - a)}{x^n + b} = \frac{x^n + b - (b - a)}{x^n + b}$$

The revised version on the right-hand side is obtained by allowing the two first occurrences of a in the numerator to cancel each other out, as we predicted they would. Our final step is to separate our expression into two rational expressions, each with the same denominator. This gives the following result:

$$\frac{x^n + b}{x^n + b} - \frac{b - a}{x^n + b}$$

Obligingly, the first term here cancels down to a value of 1, so that our original rational expression is now transformed into:

$$1 - \frac{b - a}{x^n + b}$$

We have now derived two useful formulae which can be used where needed to make certain rational expressions more amenable to the process of integration. These formulae are:

$$\frac{x^n}{x^n + a} = 1 - \frac{a}{x^n + a} \text{ and } \frac{x^n + a}{x^n + b} = 1 - \frac{b - a}{x^n + b}$$

Applications to Integration

Before we look briefly at how this algebraic method helps us to integrate expressions, let's see a couple of concrete examples of the technique.

First, a reminder of the informal use we made of this approach in [Part 2 Section 4 Example 4](#). There, we transformed a rational expression in x^2 as follows:

$$\frac{x^2}{x^2 + 1} = \frac{x^2 + 1 - 1}{x^2 + 1} = \frac{x^2 + 1}{x^2 + 1} - \frac{1}{x^2 + 1} = 1 - \frac{1}{x^2 + 1}$$

We can present an example of the second, more general form of the method as follows:

$$\frac{x + 2}{x + 4} = \frac{x + 2 + 2 - 2}{x + 4} = \frac{x + 4 - 2}{x + 4} = \frac{x + 4}{x + 4} - \frac{2}{x + 4} = 1 - \frac{2}{x + 4}$$

While in my general explanation of the technique earlier in this section I used expressions containing x^n terms, I would remark at this point that the method is most easily seen to be helpful in integration when $n = 1$ or 2 .

Taking the two concrete examples I have just set out, we can see that the method we are discussing can transform an intractable rational expression into a difference of two terms, one of which is simply 1, and the other of which is a simpler rational expression where any terms in x are confined to the denominator.

If we wish to integrate the two concrete examples of rational expressions I have just quoted, we will find the following fairly simple solutions – in the first case:

$$\int \frac{x^2}{x^2 + 1} dx = \int \left(1 - \frac{1}{x^2 + 1} \right) dx = x - \tan^{-1} x + C$$

and in the second case:

$$\int \frac{x + 2}{x + 4} dx = \int \left(1 - \frac{2}{x + 4} \right) dx = x - 2\ln|x + 4| + C$$

Extending the use of integration to the more general cases we have been using to explain this technique, we are able to write down some quite useful “standard” integrals – standard in the sense that they always work, whatever the values of the constant terms we represent here by the letters a and b .

Let's generalize from the first example. This gives us the following useful result:

$$\int \frac{x^2}{x^2 + a^2} dx = \int \left(1 - \frac{1}{x^2 + a^2} \right) dx = x - \frac{1}{a} \tan^{-1} \frac{1}{a} x + C$$

(Here I have adopted the common practice of representing the constant term in the denominator as a^2 instead of simply a . This not only matches the “squared” quality of the x term, but it also removes the need for disagreeable square root signs on the final solution! In order to integrate the rational expression with a numerator of 1, I have used the standard integral mentioned in [Appendix 4 Section 2](#).)

We can obtain another useful result, which will allow us to integrate a wide range of common rational expressions with ease, by generalizing our second concrete example. This gives the following:

$$\int \frac{x+a}{x+b} dx = \int \left(1 - \frac{b-a}{x+b}\right) dx = x - (b-a) \ln|x+b| + C$$

These results are listed in [Appendix 4 Section 2](#) as useful standard integrals.

3 Partial Fractions: The Cover-Up Method

In this section I will be concerned with a technique which can be used to produce partial fraction expansions of rational functions quite quickly – generally, more quickly than either of the two methods set out in the sections in [Part 3](#) dealing with the Method of Partial Fractions as a technique of use in integration. I concentrated in [Part 3](#) on the lengthier methods (they are sometimes described as “long-winded”) in order to give as full an insight into the workings of the partial fractions approach as could be achieved; I give a brief exposition here of the quicker “cover-up” method for the sake of completeness.

I shall first show this method in operation, using it to produce the partial fractions expansion of the rational expression in [Part 3 Section 1 Example 1](#), before giving some justification for the method by illustrating why it works at all. Finally, I will show how the cover-up method would help to make the working of [Part 3 Section 2 Example 4](#) a little easier than we found it in the main text.

Creating Partial Fraction Expansions Using the Cover-Up Method

I will illustrate the method of this section by using it in a step-by-step fashion to produce the partial fractions expansion of the rational expression:

$$\frac{6}{(x-4)(x-1)}$$

which we integrated at the start of [Part 3 Section 1](#).

Since the denominator of this expression contains two simple linear factors, our chosen expansion would be of the form:

$$\frac{A}{x-4} + \frac{B}{x-1}$$

The cover-up method consists of a set of steps which are repeated for each of the linear factors in the denominator of our original expression. These steps produce a constant as their result each time, and this constant is the value of the numerator which we write over the

linear factor in the partial fractions expansion.

The steps are as follows (starting with the first linear factor $x - 4$):

Step 1: Choose the value for x which will make the linear factor take a value of zero. This amounts to solving the equation:

$$x - 4 = 0$$

which we can see gives a solution of $x = 4$.

Step 2: We now *cover up* the linear factor we have chosen to work with, and write down the expression which remains once this factor has been hidden from view. The expression we obtain is as follows:

$$\frac{6}{x - 1}$$

Step 3: We now substitute the value of x which we obtained in Step 1 into the revised expression of Step 2, and evaluate the result. In this case we will find the following value:

$$\frac{6}{4 - 1} = \frac{6}{3} = 2$$

Step 4: We can now write down the first term in our partial fractions expansion by making the result of Step 3 the numerator of the term whose denominator is $x - 4$, our chosen linear factor/ This term will be:

$$\frac{2}{x - 4}$$

To complete the process, we will now repeat steps 1 to 4 for the second linear factor in our original denominator, which is $x - 1$. (Indeed, we would now repeat these steps for each and every linear factor which remained in our denominator in the event that it contained more than two.) I will summarise these steps more briefly in this second case.

In step 1, the solution of the equation $x - 1 = 0$ gives us a value of $x = 1$.

Covering up the factor $x - 1$ in our original expression gives the following revised expression:

$$\frac{6}{x - 4}$$

and substituting $x = 1$ into it gives us the result -2 .

The second term of our expansion is therefore:

$$\frac{-2}{x - 1}$$

Putting these two terms together gives us the final partial fractions expansion:

$$\frac{2}{x-4} - \frac{2}{x-1}$$

which was exactly the result we obtained in the main text using the method of equating numerators.

Why Does This Method Work?

In order to justify this approach to producing partial fractions expansions, I will sketch out here some ideas which will indicate why the method works. This will scarcely constitute a formal proof, although it will give some insight into why we can use this method with confidence.

Let's consider a general partial fractions expansion of a rational function. We generally expect to be able to create expansions along the following lines:

$$\frac{px + q}{(x + a)(x + b)} = \frac{A}{x + a} + \frac{B}{x + b}$$

In [Part 3 Section 3](#), I set out a chain of reasoning which shows that when we re-express the right-hand side of the above equation as a single rational expression using the common denominator $(x + a)(x + b)$, and then equate the numerator of this new expression with the numerator of the original rational expression – the equation we obtain is as follows:

$$A(x + b) + B(x + a) = px + q$$

we can then choose a value for x which will make the factor multiplied by B take a value of zero. This value here would be $x = -a$. The argument in [Part 3 Section 3](#) goes on to conclude that this gives us a simple equation from which we can obtain an expression for the value of the placeholder A . This value turns out to be the following:

$$A = \frac{q - ap}{b - a}$$

Let's now see what happened when we try applying the cover-up method to the general rational expression we started with. This expression was:

$$\frac{px + q}{(x + a)(x + b)}$$

Following the method as set out above, we first choose the value of x which will make the first linear factor in the denominator become zero. This is, of course, $x = -a$, just as we saw in the argument based on equating numerators.

We now cover up this first linear factor, obtaining the new expression:

$$\frac{px + q}{x + b}$$

and we now set x equal to our chosen value of $-a$ in this expression, which gives us the following result:

$$\frac{-ap + q}{-a + b} = \frac{q - ap}{b - a}$$

which is precisely the value for A which the method of equating numerators gave us earlier.

An exactly analogous argument can be made out for the case of the second placeholder B . So we can see that the cover-up method could in some ways be considered simply to be a consequence of the method of equating numerators. It can be argued, however, that it is a good deal quicker and simpler to carry out than the full-blown method as we have used it in the main text.

We will go one step further here, however, and give some justification for using the cover-up method to help with expansions in cases where not all of the factors in the denominator are simple linear factors. To do this, let's look at the following general rational expression:

$$\frac{px + q}{(x + a)P(x)}$$

In this expression, we will take $P(x)$ to stand for some (unspecified) polynomial in x . Bearing in mind our experiences in the main text, we might expect to create an expansion of some form like this:

$$\frac{A}{x + a} + \frac{N(x)}{P(x)}$$

Here, $N(x)$ is used to represent some numerator expression in x of whatever type is appropriate to the polynomial in the denominator. Can we use the cover-up method at least to obtain a value for the placeholder A ?

To answer that question, let's see how far we can get by looking at what the method of equating numerators gives us in this general case. Re-expressing the general expansion we set out a few lines above using the common denominator $(x + a)P(x)$ gives us the following equation of numerators:

$$AP(x) + (x + a)N(x) = px + q$$

Luckily (or not, since a linear factor is in play here!) we can choose a value for x which will make the factor $x + a$ take a zero value, and leave us with an equation of sorts in the placeholder A alone, without the disquieting presence of that unspecified numerator $N(x)$. This value is, as we might expect, $x = -a$, and it transforms our equation of numerators into the following:

$$AP(-a) = q - ap \text{ giving us } A = \frac{q - ap}{P(-a)}$$

Now let's compare this with what an experimental application of the cover-up method would give us. To start with, we note once more that a value of $x = -a$ will send the single linear factor in our denominator to zero. Covering up the linear factor gives us the following revised rational expression:

$$\frac{px + q}{P(x)}$$

and we can then substitute into this our value $x = -a$ to obtain the following:

$$\frac{-ap + q}{P(-a)} = \frac{q - ap}{P(-a)}$$

Looking back a few paragraphs will provide the happy confirmation that this is exactly what we might have hoped for, since it is the same result which the method of equating numerators produced! Therefore we will conclude that the cover-up method can be used to deal with at least part of the work of producing partial fraction expansions, whenever there are at least *some* linear factors in the denominator of the rational expression which we start with.

We conclude this look at the cover-up method with an example of how this last discovery might help us in a particularly tricky case from the main text.

How the Cover-Up Method Can Help with More Difficult Examples

In [Part 3 Section 2 Example 4](#), we wanted to obtain a partial fractions expansion of the following rational expression:

$$\frac{x - 1}{(x - 2)(x^2 - 4x + 5)^2}$$

We concluded that the expansion required would be of the following form:

$$\frac{A}{x - 2} + \frac{Bx + C}{x^2 - 4x + 5} + \frac{Dx + E}{(x^2 - 4x + 5)^2}$$

and we determined the values of the five placeholders in the numerators of the expansion by creating and solving five simultaneous equations in five variables.

As we saw in the concluding argument of the preceding subsection, we might expect to be able to use the cover-up method with this example, at least in order to obtain the value of the first placeholder A . Let's see how this would work, and what effect it would have on the overall complexity of the task of establishing values for the remaining placeholders.

To proceed through the steps of the cover-up method, then, we start by deciding what value of x will make $x - 2$, the linear factor in the original expression's denominator, take a value of zero. Of course, this value will be $x = 2$. We now proceed to cover up the linear factor $x - 2$ in the original expression, leaving us with:

$$\frac{x - 1}{(x^2 - 4x + 5)^2}$$

Finally, we substitute our chosen value $x = 2$ into this revised expression. The result, which will give us our value for A , is:

$$\frac{2 - 1}{(2^2 - 4 \cdot 2 + 5)^2} = \frac{1}{1^2} = 1$$

That was certainly simpler than solving linear equations would have been. However, this has only given us part of the solution. In order to proceed, let's write down our partial fractions expansion to the extent that we now know it:

$$\frac{1}{x - 2} + \frac{Bx + C}{x^2 - 4x + 5} + \frac{Dx + E}{(x^2 - 4x + 5)^2}$$

Let's now try to complete the job by equating numerators as we did in the main text, but with the added piece of information we now have about the value of A . Re-expressing the expansion as a single rational expression with the common denominator:

$$(x - 2)(x^2 - 4x + 5)^2$$

Our numerator will be as follows:

$$(x^2 - 4x + 5)^2 + (Bx + C)(x - 2)(x^2 - 4x + 5) + (Dx + E)(x - 2)$$

(We note that this is exactly the numerator we found at this stage in the main text, but with the value $A = 1$ put into place.)

We multiply this expression out, just as we did in the main text. I will in this instance move straight to the list of coefficients of powers of x which we obtain, so as to save the tedium of working through all the algebra once again with only one small difference. These coefficients, with their associated powers, are:

$$\begin{aligned} &(B + 1)x^4 \\ &(C - 6B - 8)x^3 \\ &(13B - 6C + D + 26)x^2 \\ &(13C - 10B + E - 2D - 40)x \\ &25 - 10C - 2E \end{aligned}$$

(These can, of course, be written down directly by starting with the list of coefficients in the main text and just substituting in the value $A = 1$, which we have already established.) The expression with these coefficients and powers of x must then be equated with the original numerator $x - 1$. This will yield the following set of equations, which we get by equating coefficients of powers of x :

$$\begin{aligned} B + 1 &= 0 \\ C - 6B - 8 &= 0 \\ 13B - 6C + D + 26 &= 0 \end{aligned}$$

$$\begin{aligned}13C - 10B + E - 2D - 40 &= 1 \\ 25 - 10C - 2E &= -1\end{aligned}$$

We have here a set of five equations, but involving only four variables – the remaining placeholders from B to E . And, just by looking at the first of these, it would seem that our task of solving them will be rather simpler than the task which faced us in the text.

Indeed, we can begin by simply solving the first equation to give us the value:

$$B = -1$$

Putting this value into the second equation, which now contains only the variables B and C . We have the following:

$$C + 6 - 8 = 0 \text{ or } C - 2 = 0 \text{ so that } C = 2$$

Remarking that the solutions are already “dropping out” very smoothly and easily, we can proceed to substitute our values for B and C into the third equation, in order to obtain this equation for the value of D :

$$-13 - 12 + D + 26 = 0 \text{ or } D + 1 = 0 \text{ so that } D = -1$$

Finally, we can see that by substituting in the value we have obtained for C , the fifth equation will now give us an easy route to the value of the final placeholder E :

$$25 - 20 - 2E = -1 \text{ or } 5 - 2E = -1 \text{ so that } 2E = 6 \text{ and } E = 3$$

Our list of the remaining placeholders and their values now runs as follows:

$$\begin{aligned}B &= -1 \\ C &= 2 \\ D &= -1 \\ E &= 3\end{aligned}$$

so that we can now write down our partial fractions expansion (the same expansion we obtained with so much labour in [Part 3 Section 2](#)) as follows:

$$\frac{1}{x-2} + \frac{2-x}{x^2-4x+5} + \frac{3-x}{(x^2-4x+5)^2}$$

or alternatively, arranging the numerators so that the x term comes first:

$$\frac{1}{x-2} - \frac{x-2}{x^2-4x+5} - \frac{x-3}{(x^2-4x+5)^2}$$

Before leaving this example, however, it cannot have escaped the reader’s notice that we haven’t made any use of the fourth of our equations involving the various placeholders in the above reasoning. In one sense, this does not surprise us: we would only expect to need four simultaneous equations to find the values of four variables. However, since the fourth

equation above arose in the same way as all the others, we ought to spend just a few moments confirming that the results we have obtained for the placeholders B to E actually satisfy the fourth equation as well.

In order to do this, we recall that the fourth equation in our set of five was as follows:

$$13C - 10B + E - 2D - 40 = 1$$

which actually contains an instance of all four of the placeholders we have just determined. Substituting the values we have found into this equation gives us the following left-hand side:

$$13 \cdot 2 - 10 \cdot (-1) + 3 - 2 \cdot (-1) - 40$$

Carrying out all the multiplications and applying the correct signs, this reduces to:

$$26 + 10 + 3 + 2 - 40 = 1$$

so we see that this equation is indeed satisfied by our values.

This was somewhat easier than the method we adopted in the main text. In particular, using the cover-up method at the start to determine the value of the placeholder A made the set of equations we obtained by equating coefficients in the numerators considerably simpler, so that the values of the remaining four placeholders could be largely “read off” after choosing the most fruitful equation to use in each case and substituting in the values of those placeholders already found.

Appendix 4

Reference Material

1 Some Standard Derivatives

Some standard derivatives, in addition to those I have considered prerequisites for this text, have been given or used in the course of the main part of this book. These are set out here for ease of reference:

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \tan x = \sec^2 x = \tan^2 x + 1$$

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{x^2 + 1}$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

2 Some Standard Integrals

The integrals listed in this section are sometimes regarded as standard integrals, and I have gathered them together here as a list for reference.

The following integrals can be regarded as more “advanced” versions of the usual integrals of the sine, cosine and exponential functions, and will certainly save some labour:

$$\int \sin kx \, dx = -\frac{1}{k} \cos kx + C$$

$$\int \cos kx \, dx = \frac{1}{k} \sin kx + C$$

$$\int e^{kx} \, dx = \frac{1}{k} e^{kx} + C$$

We quoted the next integral in connection with trigonometric substitution, and demonstrate its evaluation using substitution together with some algebraic scaffolding in [Appendix 5 Section 6](#):

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

and this integral was presented as an example in [Part 1 Section 7](#):

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \ln|\sin x| + C$$

Some of the following integrals, which are often quoted without proof as standard integrals, arose as special cases of the integrals we were trying to e in [Part 1 Section 11](#):

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

$$\int -\frac{1}{\sqrt{1-x^2}} dx = \cos^{-1} x + C$$

$$\int \frac{1}{x^2+1} dx = \tan^{-1} x + C$$

It is possible, using reasoning exactly like that used in [Part 1 Section 11](#), to generalise the above integrals to obtain the following, which are also sometimes treated as standard results:

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \frac{1}{a} x + C$$

$$\int -\frac{1}{\sqrt{a^2-x^2}} dx = \cos^{-1} \frac{1}{a} x + C$$

$$\int \frac{a}{x^2+a^2} dx = \tan^{-1} \frac{1}{a} x + C$$

In this last integral, if we bear in mind that the constant a in the numerator of the integrand can be moved outside the integral sign, we should be able to see that it can be rewritten as follows:

$$a \int \frac{1}{x^2+a^2} dx = \tan^{-1} \frac{1}{a} x + C$$

Dividing both sides by a will give us the following slightly revised version of the standard integral:

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \frac{1}{a} x + C$$

[Example 4 of Part 1 Section 11](#) shows that trigonometric substitution can be used to integrate certain square root expressions where the square root is not necessarily in the denominator of a rational function. The following standard integrals arise from this method of integration:

$$\int \sqrt{a^2-x^2} dx = \frac{1}{2} \left(a^2 \sin^{-1} \frac{x}{a} + x \sqrt{a^2-x^2} \right) + C$$

$$\int \sqrt{x^2-a^2} dx = \frac{1}{2} \left(x \sqrt{x^2-a^2} - a^2 \ln \left| x + \sqrt{x^2-a^2} \right| \right) + C$$

$$\int \sqrt{x^2 + a^2} dx = \frac{1}{2} \left(x\sqrt{x^2 + a^2} + a^2 \ln |x + \sqrt{x^2 + a^2}| \right) + C$$

The first of the three is obtained using sine-substitution, as demonstrated in [Part 1 Section 11 Example 4](#); the second responds to secant-substitution, and the third to tangent-substitution.

Two useful results were obtained in [Appendix 3 Section 2](#), using the method of conversion of rational functions. I list them here for ease of reference:

$$\int \frac{x^2}{x^2 + a^2} dx = x - \frac{1}{a} \tan^{-1} \frac{1}{a} x + C$$

$$\int \frac{x + a}{x + b} dx = x - (b - a) \ln |x + b| + C$$

In [Part 2 Section 8](#), we evaluated the following integrals by parts, employing the helpful technique of restating each of them as a function multiplied by a factor of 1:

$$\int \ln x dx = x \ln x - x + C$$

$$\int \tan^{-1} x dx = x \tan^{-1} x - \frac{1}{2} \ln(x^2 + 1) + C$$

We can use the power-reduction formulae quoted in the section on trigonometric identities below, the formula for $\tan^2 x$, and finally the standard derivative of $\tan x$ quoted above to set down the following useful results, which we may wish to regard as standard integrals for the squares of some trigonometric functions:

$$\int \sin^2 x dx = \frac{1}{2} \int (1 - \cos 2x) dx = \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) + C = \frac{1}{2} (x - \sin x \cos x) + C$$

$$\int \cos^2 x dx = \frac{1}{2} \int (1 + \cos 2x) dx = \frac{1}{2} \left(x + \frac{1}{2} \sin 2x \right) + C = \frac{1}{2} (x + \sin x \cos x) + C$$

$$\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C$$

$$\int \sec^2 x dx = \tan x + C$$

The second versions of the integrals of $\sin^2 x$ and $\cos^2 x$ are obtained from the first version in each case by applying the double-angle identity for $\sin 2x$ - this is listed below in the section on trigonometric identities.

The Integral of an Irrational Function

At the end of [Part 3 Section 7](#), we used the integration of a difficult-looking irrational function to obtain the following standard integral, which is similar in its usefulness to those

involving the square roots of quadratic denominators which we listed in the preceding discussion of standard integrals . The integral was:

$$\int \frac{1}{\sqrt{x^2 + 1}} dx = \ln |x + \sqrt{x^2 + 1}| + C$$

I gave an explicit demonstration of the evaluation of this integral in the note following the final example in [Part 3 Section 7](#).

A more general standard integral, of which the above one is merely a particular instance, is as follows:

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln |x + \sqrt{x^2 + a^2}| + C$$

This general version is demonstrated in detail in [Appendix 5 Section 20](#).

I have listed this additional standard integral here since it serves to round out the list of integrals with irrational denominators which were quoted in the preceding subsection.

Integrals of Some Useful Rational Functions

In [Part 2 Section 7](#), we illustrated our rule for choosing parts when integrating by parts with some examples. These required us to integrate various rational functions. I list here the integrals of some useful rational functions of the kind we dealt with, giving a brief indication of how we evaluated their integrals.

$$\int \frac{x}{x^2 + 1} dx = \frac{1}{2} \ln(x^2 + 1) + C$$

(This is carried out by substitution, and a sketch of the process appears in [Part 2 Section 7 Example 2](#), where we integrated this function multiplied by a factor of 2.)

$$\int \frac{x^2}{x^2 + 1} dx = x - \tan^{-1} x + C$$

(This was integrated by parts in [Part 2 Section 4 Example 4](#), and by tangent-substitution in [Appendix 5 Section 8](#).)

$$\int \frac{x^3}{x^2 + 1} dx = \frac{1}{2} (x^2 - \ln(x^2 + 1)) + C$$

(This was evaluated using substitution in [Part 1 Section 8 Example 5](#).)

Standard Integrals Arising from the Method of Partial Fractions

[Part 3 Section 1](#) explains and illustrates the application to integration of the Method of Partial Fractions. Two useful standard integrals obtained using this method are quoted in that section, and a full derivation of each is given in [Appendix 5 Section 16](#).

These two rules are listed for reference here:

$$\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{a-b} \ln \left| \frac{x+b}{x+a} \right| + C$$

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

A Rule for Integrating Some Rational Functions with a Quadratic Denominator

In [Part 3 Section 5](#), we learnt to apply polynomial division in order to help us integrate some rational functions with linear or quadratic denominators. For some of those with quadratic denominators, it is possible to integrate the remainder term which is left over after the polynomial division using a rule of integration which is derived in [Appendix 5 Section 19](#).

This rule is as follows:

$$\int \frac{px + q}{x^2 + bx + c} dx = \frac{p}{2} \ln|x^2 + bx + c| - \frac{bp - 2q}{\sqrt{4c - b^2}} \tan^{-1} \frac{2}{\sqrt{4c - b^2}} \left(x + \frac{b}{2} \right) + C$$

There is a limitation on the applicability of this rule, however, due to the presence of square roots in the denominators of two of its coefficients. The rule can only be applied when the expression under the square root signs is positive. We can therefore only use this rule in cases where the coefficients in the integrand satisfy the following inequality:

$$4c - b^2 > 0$$

This is equivalent to the inequality:

$$b^2 - 4c < 0$$

which is the condition that the quadratic in the denominator should have no real roots.

We see from this that this formula only applies if the denominator in the integrand is an *irreducible* quadratic. Integrals of this type whose denominator can be factorised should be dealt with using partial fractions once the denominator has been expressed as the product of two linear factors.

Standard Integrals Derived from the Product-to-Sum Formulae

The product-to-sum formulae for trigonometric functions are listed in the following section on trigonometric identities. Since this section is intended to catalogue the various standard integrals we have encountered in the text, however, I will list the three integrals which I have derived from these formulae here. They are as follows:

$$\int \sin mx \sin nx dx = \frac{1}{2(m-n)} \sin(m-n)x - \frac{1}{2(m+n)} \sin(m+n)x + C$$

$$\int \cos mx \cos nx \, dx = \frac{1}{2(m-n)} \sin(m-n)x + \frac{1}{2(m+n)} \sin(m+n)x + C$$

$$\int \sin mx \cos nx \, dx = -\frac{1}{2(m+n)} \cos(m+n)x - \frac{1}{2(m-n)} \cos(m-n)x + C$$

Some Rules for Integration

In [Part 1 Section 7](#), we used the knowledge we had gained about the technique of integration by substitution to derive the following rules for writing down solutions to some integrals without having to work through a particular technique:

There are two general rules:

$$\int (f(x))^n f'(x) \, dx = \frac{1}{n+1} (f(x))^{n+1} + C$$

$$\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$$

Consequences of these allow us to integrate quickly some functions containing linear expressions, as follows:

$$\int (ax+b)^n \, dx = \frac{1}{a(n+1)} (ax+b)^{n+1} + C$$

$$\int \frac{1}{ax+b} \, dx = \frac{1}{a} \ln|ax+b| + C$$

If the linear expressions are even simpler, we can use the following two rules:

$$\int (x+k)^n \, dx = \frac{1}{n+1} (x+k)^{n+1} + C$$

$$\int \frac{1}{x+k} \, dx = \ln|x+k| + C$$

Finally, we have the following useful rules which apply when we are integrating an expression comprising any function $f(x)$. We adopt the following convention for the integral of $f(x)$:

$$\int f(x) \, dx = F(x) + C$$

Our two rules are as follows:

$$\int f(x+k) \, dx = F(x+k) + C$$

$$\int f(kx)dx = \frac{1}{k}F(kx) + C$$

These two rules may be combined to obtain the more general version:

$$\int f(ax + b)dx = \frac{1}{a}F(ax + b) + C$$

The General Form of Integration by Substitution

We can use the general reasoning set out in [Part 1 Section 5](#) to write down the equation which shows the effect of straightforward u -substitution on integrals of the appropriate type. It is as follows:

$$\int kg'(f(x))f'(x)dx = kg(f(x)) + C$$

This can usefully be considered a “rule of integration” in itself – if we can decompose an integrand and allocate the various components of the expressions on each side of the above equation to parts of the integrand we are working with – and these parts would be as follows:

$$k, f(x), f'(x), g(x), g'(x) \text{ and } g'(f(x))$$

then we can use the above equation to write down the result of the integration easily without going through the whole process of substitution explicitly. A number of examples of this process in action are given in [Part 1 Section 5](#), and I have used the method from time to time elsewhere in this book so as to avoid working through the substitution process in full.

It is important to note that this rule will only yield an immediate result if it is easy to deduce $g(x)$ from our expression for $g'(x)$. In other words, we must be able to integrate $g'(x)$ easily. If this task requires the application of some specific technique, then the rule is still useful as a first stage, but will not give us an immediate result which we can write down straight away.

A Rule of Integration Arising from Integration by Parts

At the very end of [Part 3 Section 9](#), I proposed a rule of integration which might help save some work when integrating expressions of the form $xf(x)$, where $f(x)$ represented one of the combination trigonometric functions examined at length in that section.

The rule, which does actually have more general application, can be stated as follows:

$$\int xf(x)dx = xF(x) - \int F(x)dx \text{ where } F(x) = \int f(x)dx$$

More generally, it can also be applied where $f(x)$ represents a function which can be classified as exponential for the purposes of the **LIATE** rule which we use to choose parts when applying integration by parts.

In order to be able to use this rule at all, however, we need to be able to integrate the function $F(x)$ which appears as part of the left-hand side integrand.

Reduction Formulae

We have derived several reduction formulae for the integrals of powers of functions in the text, particularly in [Part 2](#).

Here is a list of these, gathered together for convenient reference:

$$I_n = \int \sin^n x dx \text{ gives the reduction formula } I_n = \frac{n-1}{n} I_{n-2} - \frac{1}{n} \sin^{n-1} x \cos x$$

$$I_n = \int \cos^n x dx \text{ gives the reduction formula } I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n} \sin x \cos^{n-1} x$$

$$I_n = \int \sec^n x dx \text{ gives the reduction formula } I_n = \frac{1}{n-1} \sec^{n-2} x \tan x - \frac{n-2}{n-1} I_{n-2}$$

$$I_n = \int \ln^n x dx \text{ gives the reduction formula } I_n = x \ln^n x - n I_{n-1}$$

Integrals of Powers of Trigonometric Functions

In [Part 2 Section 13](#), we derived reduction formulae for the integrals of the n -th powers of $\sin x$ and $\cos x$. Since we can easily integrate these functions themselves, and the power-reduction formulae allow us to integrate their squares, we can clearly use the reduction formulae given in the text to obtain integrals for the first few higher powers of these functions.

The tables on the following pages set out the integrals of the first six powers of the sine, cosine and secant functions, using what we know about their lowest powers, and using the reduction formulae to construct integrals of the higher powers based on these.

<i>Integral</i>	<i>Reduction Formula and Solution</i>
$I_1 = \int \sin x \, dx$	$I_1 = -\cos x + C$
$I_2 = \int \sin^2 x \, dx$	$I_2 = \frac{1}{2}x - \frac{1}{2}\sin x \cos x + C$
$I_3 = \int \sin^3 x \, dx$	$I_3 = \frac{2}{3}I_1 - \frac{1}{3}\sin^2 x \cos x$ $I_3 = -\frac{2}{3}\cos x - \frac{1}{3}\sin^2 x \cos x + C$
$I_4 = \int \sin^4 x \, dx$	$I_4 = \frac{3}{4}I_2 - \frac{1}{4}\sin^3 x \cos x$ $I_4 = \frac{3}{8}x - \frac{3}{8}\sin x \cos x - \frac{1}{4}\sin^3 x \cos x + C$
$I_5 = \int \sin^5 x \, dx$	$I_5 = \frac{4}{5}I_3 - \frac{1}{5}\sin^4 x \cos x$ $I_5 = -\frac{8}{15}\cos x - \frac{4}{15}\sin^2 x \cos x - \frac{1}{5}\sin^4 x \cos x + C$
$I_6 = \int \sin^6 x \, dx$	$I_6 = \frac{5}{6}I_4 - \frac{1}{6}\sin^5 x \cos x$ $I_6 = \frac{15}{48}x - \frac{15}{48}\sin x \cos x - \frac{5}{24}\sin^3 x \cos x - \frac{1}{6}\sin^5 x \cos x + C$

<i>Integral</i>	<i>Reduction Formula and Solution</i>
$I_1 = \int \cos x \, dx$	$I_1 = \sin x + C$
$I_2 = \int \cos^2 x \, dx$	$I_2 = \frac{1}{2}x + \frac{1}{2}\sin x \cos x + C$
$I_3 = \int \cos^3 x \, dx$	$I_3 = \frac{2}{3}I_1 + \frac{1}{3}\sin x \cos^2 x$ $I_3 = \frac{2}{3}\sin x + \frac{1}{3}\sin x \cos^2 x + C$
$I_4 = \int \cos^4 x \, dx$	$I_4 = \frac{3}{4}I_2 + \frac{1}{4}\sin x \cos^3 x$ $I_4 = \frac{3}{8}x + \frac{3}{8}\sin x \cos x + \frac{1}{4}\sin x \cos^3 x + C$
$I_5 = \int \cos^5 x \, dx$	$I_5 = \frac{4}{5}I_3 + \frac{1}{5}\sin x \cos^4 x$ $I_5 = \frac{8}{15}\sin x + \frac{4}{15}\sin x \cos^2 x + \frac{1}{5}\sin x \cos^4 x + C$
$I_6 = \int \cos^6 x \, dx$	$I_6 = \frac{5}{6}I_4 + \frac{1}{6}\sin x \cos^5 x$ $I_6 = \frac{15}{48}x + \frac{15}{48}\sin x \cos x + \frac{5}{24}\sin x \cos^3 x + \frac{1}{6}\sin x \cos^5 x + C$

<i>Integral</i>	<i>Reduction Formula and Solution</i>
$I_1 = \int \sec x \, dx$	$I_1 = \ln \sec x + \tan x + C$
$I_2 = \int \sec^2 x \, dx$	$I_2 = \tan x + C$

$I_3 = \int \sec^3 x dx$	$I_3 = \frac{1}{2} \sec x \tan x - \frac{1}{2} I_1$ $I_3 = \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln \sec x + \tan x + C$
$I_4 = \int \sec^4 x dx$	$I_4 = \frac{1}{3} \sec^2 x \tan x - \frac{2}{3} I_2$ $I_4 = \frac{1}{3} \sec^2 x \tan x - \frac{2}{3} \tan x + C$
$I_5 = \int \sec^5 x dx$	$I_5 = \frac{1}{4} \sec^3 x \tan x - \frac{3}{8} \sec x \tan x + \frac{3}{8} \ln \sec x + \tan x + C$
$I_6 = \int \sec^6 x dx$	$I_6 = \frac{1}{5} \sec^4 x \tan x - \frac{4}{15} \sec^2 x \tan x - \frac{8}{15} \tan x + C$

3 Some Trigonometric Identities

We have frequently quoted and used the Pythagorean identity connecting the sine and cosine, as well as the associated identity which connects tangent and secant. These identities are as follows:

$$\sin^2 x + \cos^2 x = 1$$

$$\sec^2 x - \tan^2 x = 1$$

Just like the Pythagorean identity, the second of these two identities has two possible rearrangements; these both give useful equivalences which can be of great help to us when integrating. I list them explicitly since they may be less familiar than the common rearrangements of the Pythagorean identity:

$$\tan^2 x = \sec^2 x - 1$$

$$\sec^2 x = \tan^2 x + 1$$

We have also used the double-angle formulae, which help us to rewrite expressions involving sines and cosines of multiples of x :

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

Some rearrangements of the double-angle cosine formula (derived with the help of the Pythagorean identity to express $\cos 2x$ in terms first of the cosine only, and then of the sine only) are worth noting:

$$\cos 2x = 2\cos^2 x - 1$$

$$\cos 2x = 1 - 2\sin^2 x$$

Reference Material

Finally, we have sometimes made use of the power-reduction formulae, which allow us to transform the squares of the sine and cosine into a more manageable form – particularly for integration. These are as follows:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

The Addition Formulae

The following trigonometric identities are often learned quite early in our exploration of trigonometry. They don't really have much immediate application to integration, but I am quoting them here because they can be used to produce some much more useful formulae, which will be listed presently.

The most common addition formulae are these:

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

The Product-to-Sum Formulae

If we add together the left-hand sides of selected pairs of the above formulae, then we will obtain some new formulae which will allow us to convert products of trigonometric functions into a more manageable form.

For example, if we consider the expression:

$$\sin(\alpha + \beta) + \sin(\alpha - \beta)$$

then the above addition formulae allow us to see that this is equivalent to:

$$2\sin \alpha \cos \beta$$

Similar combinations of the left-hand sides of the addition formulae will lead us to the following set of product-to-sum formulae:

$$\sin \alpha \sin \beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$$

$$\cos \alpha \cos \beta = \frac{1}{2}(\cos(\alpha - \beta) + \cos(\alpha + \beta))$$

$$\sin \alpha \cos \beta = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta))$$

$$\cos \alpha \sin \beta = \frac{1}{2}(\sin(\alpha + \beta) - \sin(\alpha - \beta))$$

The last two of these identities can be considered alternative versions of the formula for the product of a sine and a cosine – the difference is simply the result of the choices we make as to which function takes the argument α and which takes the argument β .

These are much more immediately useful to us than the addition formulae from which they were derived, since they transform hard-to-integrate products of trigonometric functions into simple sums which are often quite easy to integrate.

For example, if we consider the following product of trigonometric functions, which we might consider rather hard to integrate:

$$\sin mx \cos nx$$

we will find that the third of the above formulae will transform the expression into:

$$\frac{1}{2}(\sin(m+n)x + \sin(m-n)x)$$

or, in abbreviated form:

$$\frac{1}{2}(\sin Mx + \sin Nx)$$

where $M = m + n$ and $N = m - n$. This latter expression could be integrated easily.

4 Compounds of Trigonometric Functions and Inverse Trigonometric Functions

We often encounter compound functions of this kind in evaluating integrals by means of trigonometric substitution. Solutions to integrals of this type can often be rewritten in a manner more in keeping with the integrand we began with by using some of the following identities, which can be derived using the Pythagorean identities we set out at the start of this section. After listing the more common examples of such identities, I give derivations of these below.

$$\sec \tan^{-1} x = \sqrt{x^2 + 1}$$

$$\tan \sec^{-1} x = \sqrt{x^2 - 1}$$

$$\cos \sin^{-1} x = \sin \cos^{-1} x = \sqrt{1 - x^2}$$

Derivation of $\sec \tan^{-1} x = \sqrt{x^2 + 1}$

We will use the variables u and v in this derivation, noting that our approach will be to use a technique not totally dissimilar to integration by substitution. We will start by making the assignment:

$$v = \tan^{-1} u \text{ so that } u = \tan v$$

We can square both sides of the expression for u which we derived so as to obtain:

$$u^2 = \tan^2 v = \sec^2 v - 1$$

where the final right-hand side expression arises from a use of the standard trigonometric identity linking $\sec x$ and $\tan x$. A simple rearrangement allows us to say that:

$$\sec^2 v = u^2 + 1 \text{ so that } \sec v = \sqrt{u^2 + 1}$$

We are now almost finished – all that remains is to substitute back into the above identity for $\sec v$ our original definition for v which came right at the start of this chain of reasoning. This gives us the satisfying (but perhaps surprising) result that:

$$\sec \tan^{-1} u = \sqrt{u^2 + 1}$$

Here we see the combination of a trigonometric function and an inverse trigonometric function reduced to an expression completely devoid of trigonometric functions! It can be seen from the way I justified some of the algebra above that this stems directly from the identity which connects $\sec x$ and $\tan x$.

Another way of looking at this result would be to remark that the combination function $\sec \tan^{-1} x$ means "*the secant of the angle whose tangent is x* ": in other words, given the tangent of an angle (that tangent is x) the function has as its value the secant of that angle. Since the squares of $\tan x$ and $\sec x$ are related by a trigonometric identity of a Pythagorean derivation, it ought not to come as a complete surprise that, given the tangent of an angle, we should be able to compute its secant using a straightforward formula which has a distinctly Pythagorean look t it.

Derivation of $\tan \sec^{-1} x = \sqrt{x^2 - 1}$

Once again, we will use the variables u and v in the derivation. In a similar way also, we will start by assigning:

$$v = \sec^{-1} u \text{ so that } u = \sec v$$

Squaring both sides of the expression for u , we obtain:

$$u^2 = \sec^2 v = \tan^2 v + 1$$

where the final right-hand side comes from the same trigonometric identity as in the previous derivation. We can rearrange the equation we have obtained to get:

$$\tan^2 v = u^2 - 1 \text{ so that } \tan v = \sqrt{u^2 - 1}$$

Finally, substituting back our original definition for v , we have the following equivalence:

$$\tan \sec^{-1} u = \sqrt{u^2 - 1}$$

Derivation of $\cos \sin^{-1} x = \sqrt{1 - x^2}$

In order to demonstrate this result, let's start as we did before, using variables u and v and first setting

$$v = \sin^{-1} u \text{ so that } u = \sin v$$

Squaring both sides of the expression for u , we have:

$$u^2 = \sin^2 v = 1 - \cos^2 v$$

using the Pythagorean identity which links the squares of $\cos x$ and $\sin x$. Rearranging this equation gives us:

$$\cos^2 v = 1 - u^2 \text{ so that } \cos v = \sqrt{1 - u^2}$$

Finally, substituting back our original definition for v gives the following equivalence:

$$\cos \sin^{-1} u = \sqrt{1 - u^2}$$

Derivation of $\sin \cos^{-1} x = \sqrt{1 - x^2}$

Once again, and in a very similar way to the preceding derivation, we begin by assigning:

$$v = \cos^{-1} u \text{ so that } u = \cos v$$

Squaring both sides of the expression for u gives us:

$$u^2 = \cos^2 v = 1 - \sin^2 v$$

Rearrangement of this equation gives the following:

$$\sin^2 v = 1 - u^2 \text{ so that } \sin v = \sqrt{1 - u^2}$$

Finally, substituting back our original definition for v gives the following equivalence:

$$\sin \cos^{-1} u = \sqrt{1 - u^2}$$

We note, I hope with some interest, that the square-root expressions for $\cos \sin^{-1} x$ and $\sin \cos^{-1} x$ are exactly the same. This arises because the functions $\sin x$ and $\cos x$ behave with perfect symmetry as regards the Pythagorean identity which links them – in other words,

they can be swapped around without changing the identity at all. This is, however, not true for the identity which links $\tan x$ and $\sec x$, which we used in the first two derivations – these each led to slightly different square-root expressions.

A More Advanced Example

We encountered one rather more complicated combination of trigonometric and inverse trigonometric functions in the text. The identity we had to use was as follows:

$$\sin 2 \tan^{-1} x = \frac{2x}{x^2 + 1}$$

Derivation of $\sin 2 \tan^{-1} x = 2x/(x^2 + 1)$

We will start by setting:

$$v = \tan^{-1} u \text{ so that } u = \tan v$$

Before proceeding, we will recall the trigonometric identity which allows us to state in the current context that:

$$\sec^2 v = \tan^2 v + 1$$

Given the relationship between u and v which we have already set out, we can rewrite this as follows:

$$\sec^2 v = \tan^2 v + 1 = u^2 + 1$$

Since we know (by definition) that:

$$\sec v = \frac{1}{\cos v} \text{ so that } \sec^2 v = \frac{1}{\cos^2 v} \text{ and thus } \cos^2 v = \frac{1}{\sec^2 v}$$

we can confidently say that in our current situation:

$$\cos^2 v = \frac{1}{\sec^2 v} = \frac{1}{u^2 + 1} \text{ and so } \cos v = \frac{1}{\sqrt{u^2 + 1}}$$

We will require (as you will shortly see) a similar expression for $\sin^2 v$. We can obtain this by once more quoting a well-known trigonometric identity (*almost* a definition) to the effect that:

$$\tan v = \frac{\sin v}{\cos v} \text{ so that } \sin v = \tan v \cos v$$

At the start of this reasoning, we noted that $u = \tan v$. Combining this with the formula for $\cos v$ which we have just obtained, we can now state that:

$$\sin v = \tan v \cos v = \frac{u}{\sqrt{u^2 + 1}}$$

Reference Material

This was all by way of preparation. We can now get to the heart of the matter by considering what this allows us to say about $\sin 2 \tan^{-1} u$. The first thing we can say, based on our initial definition, is that:

$$\sin 2 \tan^{-1} u = \sin 2v$$

There is an alternative way of writing the right-hand side here, using the trigonometric double-angle formula for the sine. This states that:

$$\sin 2x = 2 \sin x \cos x$$

So that in our current case:

$$\sin 2v = 2 \sin v \cos v$$

Using the expressions for $\sin v$ and $\cos v$ in terms of u which we have just obtained, this means the following:

$$\sin 2v = 2 \cdot \frac{u}{\sqrt{u^2 + 1}} \cdot \frac{1}{\sqrt{u^2 + 1}}$$

Substituting back into this our original definition of v , which was $v = \tan^{-1} u$, and multiplying out the right-hand side then gives us our final simplification, which is as follows:

$$\sin 2 \tan^{-1} u = \frac{2u}{u^2 + 1}$$

5 The Integration by Parts Formula Using Differentials

A version of the integration by parts formula which differs slightly from the version I have used almost exclusively in this book can sometimes be encountered, especially in some older textbooks. I will give a brief account of this alternative version here, and explain how it corresponds to my own preferred version.

The version of the formula you may encounter is as follows:

$$\int u dv = uv - \int v du$$

This is certainly somewhat more concise than the formula I have generally used, which we will recall runs as follows:

$$\int u \frac{dv}{dx} dx = uv - \int \frac{du}{dx} v dx$$

Unless you have been well trained in the use and interpretation of the “differential” notation, you may well be asking what the alternative formula actually means, and how it is related to the one I have used.

Reference Material

Well, the first remark we can make is that the alternative formula can be “derived” from the one I have used if we are prepared to treat expressions for derivatives rather as if they were fractions. Taking this approach to the material inside each of the integral signs, we find that:

$$u \frac{dv}{dx} dx \text{ reduces to } u dv$$

by cancelling the dx terms. Doing the same thing on the right-hand side of the formula, we find that:

$$\frac{du}{dx} v dx \text{ reduces to } v du$$

by cancelling as before, then rearranging the result to put the du symbol at the end.

So much for the correspondence between the shortened formula and the more usual, fully expanded, version; there still remains the question of interpretation of the shortened version.

In order to demonstrate how this might be done, let's consider how it might be applied to [Example 1 from Part 2 Section 4](#). There we considered the integral:

$$I = \int x e^x dx$$

In order to apply the shortened version of the formula, we have to put the integrand into the form $u dv$. As before, we will set $u = x$. To complete the second part of the task, we note that because we can set:

$$\frac{dv}{dx} = e^x$$

we can therefore say (and here is the key point) that:

$$dv = e^x dx$$

By this rather roundabout reasoning, we can feel we have some justification for writing the equivalence:

$$u dv = x e^x dx$$

(Of course, someone who has been trained in the process of creating “differentials” would be able to leap directly to the above expression for dv without explicitly writing down dv/dx first.)

Moving to the right-hand side of the shortened integration by parts formula, we can obtain an expression for v by noting that:

$$dv = e^x dx \text{ so that } \frac{dv}{dx} = e^x \text{ and therefore } v = e^x$$

What about the right-hand side integral? Well, we already have a perfectly usable expression for v , obtained just above, so we are left only with the du part.. A few paragraphs earlier, we set $u = x$. This allows us to deduce that:

$$\frac{du}{dx} = 1 \text{ so that } du = 1 \cdot dx \text{ or more concisely } du = dx$$

We can therefore perhaps persuade ourselves that it is reasonable to write:

$$vdu = e^x dx$$

All this is intended to show that we can use the shortened integration by parts formula to write:

$$\int x e^x dx = x e^x - \int e^x dx$$

which is the same result we produced using the “standard” integration by parts formula in [Part 2](#) of this book.

It is probably quite safe to say that the explanation I have given here for this shortened version of the integration by parts formula has relatively little in common with the way the formula and its use might have been taught when this version was in common use. My aim here has been to connect it to the version of the formula I have chosen to use.

I have chosen the version of the formula used in [Part 2](#) since it seems much easier to explain and illustrate than the shortened formula, and takes due account of the fact that not many students engaged in learning how to integrate will have the close familiarity with the concept of “differentials” which might have been expected in the past.

Appendix 5

Supplementary Examples and Investigations

This appendix contains additional worked examples which are provided for reference, and are presented here so as to avoid interrupting the development of the main text. They include examples of the derivation of standard integrals using the techniques explained in the text, some general formulae for particular integrals which are demonstrated using symbols, and some examples of the integration of examples included in the text by a different method in order to allow an informed comparison of techniques to be made.

Some of the following examples will involve more abstract symbol-manipulation than was used in the main text, and some potentially more challenging arguments may be used in some cases.

The examples are presented in the order in which references to them appear in the main text.

1 Integration of $(x + 1)\sqrt{(x + 2)}$ Substituting $u = x + 2$.

In [Part 1 Section 8 Example 3](#), we wanted to evaluate the indefinite integral:

$$I = \int (x + 1)\sqrt{(x + 2)}dx$$

and we chose to approach this by means of a substitution which was written as an expression for x in terms of u , the opposite way round to that which we had used before. This approach was chosen in order to illustrate this “reverse-notation” approach to substitution, and to let us approach the problem in such a way as to remove fractional powers from consideration.

On this occasion, however, we will firmly decline to be put off by fractional powers, and we will instead evaluate the integral using the more “traditional” substitution of $u = x + 2$.

Indeed, we will start proceedings by immediately rewriting the original integral using the fractional-power notation, as follows:

$$I = \int (x + 1)(x + 2)^{1/2}dx$$

With this done, and noting in passing that the simple substitution we have chosen gives us the equally simple equivalence $dx = du$, we can proceed to make the substitution, which will give us:

$$\int (u - 1)u^{1/2}du = \int (u^{3/2} - u^{1/2})du$$

Integrating this using our usual rules for integrating simple combinations of powers (fractional or otherwise) term by term, we arrive at the following solution in terms of u :

$$\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$$

Reversing the substitution to obtain a solution in terms of x gives us:

$$\frac{2}{5}(x+2)^{5/2} - \frac{2}{3}(x+2)^{3/2} + C = 2(x+2)^{3/2} \left(\frac{1}{5}(x+2) - \frac{1}{3} \right) + C$$

(The second version of this solution was obtained from the first version by factorising out $2(x+2)^{3/2}$, as might be imagined from the appearance of the second version.)

To finish things off, we can multiply out the second bracketed term just as we did in [Part 1](#) to obtain the following:

$$\frac{1}{5}(x+2) - \frac{1}{3} = \frac{1}{5}x + \frac{2}{5} - \frac{1}{3} = \frac{1}{5}x + \frac{1}{15}$$

and put back our square root sign to obtain the very same final solution of:

$$\frac{2}{5}\sqrt{(x+2)^3} \left(x + \frac{1}{3} \right) + C$$

The reader may like to consider whether this method is any more or less difficult than the one we used in the body of the text.

2 Integration of $(x+a)(x+b)^p$ Using the Substitution $u = x+b$

In [Part 1 Section 8](#), I repeatedly made the point that we should expect to be able to evaluate any indefinite integral of the form:

$$I = \int (x+a)(x+b)^p dx$$

using the substitution $u = x+b$.

I propose here to give a solution to this general integral. Be prepared, however, for some more abstract symbol-manipulation than we have encountered in the concrete cases which were discussed in the text.

Moving straight into action, we know that the kind of substitution I have proposed gives us the simple equivalence $dx = du$, so that we can make our substitution and transform our integral into the following form:

$$\int (u+a-b)u^p du = \int (u^{p+1} + (a-b)u^p) du$$

We can integrate this simple sum of two powers of u using our standard rules as follows:

$$\frac{1}{p+2}u^{p+2} + \frac{a-b}{p+1}u^{p+1} + C = u^{p+1} \left(\frac{1}{p+2}u + \frac{a-b}{p+1} \right) + C$$

Reversing our substitution, we arrive at the general result in terms of x :

$$(x + b)^{p+1} \left(\frac{1}{p+2} (x + b) + \frac{a-b}{p+1} \right) + C$$

Any further simplification will depend on the arithmetical relationships between a , b and p in particular concrete cases.

3 Integration of $(ax + b)(cx + d)^p$ Using the Substitution $u = cx + d$.

This is the more general version of the integration we performed in the previous section. I claimed in [Part 1 Section 8](#) without any explicit justification the generalisation that we could evaluate the following integral:

$$I = \int (ax + b)(cx + d)^p dx$$

using the substitution technique. Just as I did in [Section 2](#) above, I will set out the various stages of the general solution to this integration problem. As was the case above, we can expect to have to grit our teeth through some careful manipulation of symbols.

We have already declared our intention of using the substitution $u = cx + d$, in order to make the bracketed term which is raised to the power p more easy to handle. We know from the note following [Part 1 Section 3 Example 5](#) that this substitution will give us the very simple equivalence:

$$dx = \frac{1}{c} du$$

In order to complete the transformation of our integral into a new integral in terms of u , we must address the question of expressing the first bracketed factor $ax + b$ in terms of u . Fortunately, despite the fact that we are dealing with symbols and not numbers, this is not too difficult.

We can first rearrange our substitution expression to express x in terms of u . This gives us:

$$x = \frac{u - d}{c}$$

so that $ax + b$ can now be rewritten as:

$$\frac{a(u - d)}{c} + b = \frac{a}{c}u + b - \frac{d}{c}$$

(From the expressions we have derived so far, we are, of course, obliged to point out that the reasoning I have already given, along with what follows, requires that c should be non-zero, in order for division by c to make any sense. Note, however, that if $c = 0$, then we can simply multiply out our first bracketed factor by d^p and integrate term by term. So this requirement on c should cause us no especial difficulty, nor should it significantly reduce the generality of our result, when it arrives.)

On making our substitution, then, our integral is transformed into the following:

$$\int \left(\frac{a}{c}u + b - \frac{d}{c} \right) u^p \frac{1}{c} du = \frac{1}{c} \int \left(\frac{a}{c}u^{p+1} + \left(b - \frac{d}{c} \right) u^p \right) du$$

Now we should note carefully that, apart from a sprinkling of additional constants, which arises from their additional presence in the integrand with which we started, this is really no more difficult to integrate than was the expression in u which we obtained in [Section 2](#) above – it is a simple sum of two powers of u .

So, armed with a determination not to succumb to a case of symbol-sickness, we can integrate with respect to u and then multiply out using the factor of $1/c$ sitting outside the integral sign to obtain the following result:

$$\frac{a}{c^2(p+2)}u^{p+2} + \left(b - \frac{d}{c} \right) \frac{1}{c(p+1)}u^{p+1} + C$$

In the interests both of greater neatness and of saving paper, we can rearrange the coefficient of u^{p+1} as follows:

$$\frac{b}{c(p+1)} - \frac{d}{c^2(p+1)} = \frac{bc}{c^2(p+1)} - \frac{d}{c^2(p+1)} = \frac{bc-d}{c^2(p+1)}$$

Finally (for the moment), the same simple factorisation step we took in [Section 2](#) above gives us this result in terms of u :

$$\frac{1}{c^2}u^{p+1} \left(\frac{a}{p+2}u + \frac{bc-d}{p+1} \right) + C$$

All that remains is to reverse the substitution in order to arrive at the following general result in terms of x :

$$\frac{1}{c^2}(cx+d)^{p+1} \left(\frac{a}{p+2}(cx+d) + \frac{bc-d}{p+1} \right) + C$$

As in [Section 2](#) above, this is probably about as far as we can usefully simplify without knowing more about the relationships between the constants in particular concrete cases.

We can allow ourselves some emotional reinforcement after this strenuous business by observing that if we set $a = 1$ and $c = 1$, then this result will reduce nicely to the result we obtained in [Section 2](#), apart from the differences in the letters used for the constant terms inside the bracketed factors.

4 Integration of $ax^{2n-1}/(bx^n + c)$ using Substitution

In the remarks which followed [Part 1 Section 8 Example 5](#), I stated that the integral:

$$I = \int \frac{ax^{2n-1}}{bx^n + c} dx$$

could be evaluated using substitution. I will set out the procedure here.

First, in line with our approach to the example from [Part 1 Section 8](#) mentioned above, we will use the substitution:

$$u = bx^n + c$$

On differentiation, this gives us:

$$\frac{du}{dx} = bnx^{n-1} \text{ so that } dx = \frac{1}{bnx^{n-1}} du$$

Making the substitution and using the expression for dx which we have just obtained, our integral is transformed into:

$$\int \frac{ax^{2n-1}}{u \cdot bnx^{n-1}} du$$

We can perform some cancellation on the x terms here, using the laws of indices to remark that:

$$\frac{x^{2n-1}}{x^{n-1}} = x^{(2n-1)-(n-1)}$$

The expression which we see here as the power of x can be simplified as follows:

$$(2n - 1) - (n - 1) = 2n - 1 - n + 1 = n$$

This brings our transformed integral into the following form:

$$\int \frac{ax^n}{bnu} du$$

In order to remove the remaining x term, we just need to note the following rearrangement of our substitution formula:

$$bx^n = u - c \text{ so that } x^n = \frac{u - c}{b}$$

We can now restate our transformed integral without any terms in x as follows:

$$\int \frac{a(u - c)}{b^2nu} du = \frac{a}{b^2n} \int \frac{u - c}{u} du$$

We have here taken the constant factor outside the integral sign, leaving us with a rather less cluttered integral in terms of u to evaluate. This integral can itself be simplified as follows:

$$\int \frac{u - c}{u} du = \int \left(\frac{u}{u} - \frac{c}{u} \right) du = \int 1 du - c \int \frac{1}{u} du$$

The integrals in the final right-hand side expression produce the following solution:

$$u - c \ln|u| + C$$

We must, of course, restore the constant factor we moved outside our original integral in terms of u to obtain the following:

$$\frac{a}{b^2n}(u - c \ln|u|) + C$$

Reversing the substitution, we now reach the following solution in terms of x :

$$\frac{a}{b^2n}(bx^n + c - c \ln|bx^n + c|) + C$$

This can be tidied up a little further by restating the constant of integration as follows:

$$C' = C + \frac{ac}{b^2n}$$

This operation removes the explicit constant term from the solution, giving us the following final solution in terms of x :

$$\frac{a}{b^2n}(bx^n - c \ln|bx^n + c|) + C'$$

(One concluding remark: if we set a and b in our original integral to a value of 1, and substitute a for c , we can see that the general solution we have obtained here gives us the solution to [Part 1 Section 8 Example 5](#) as a special case.)

5 A Comparison of Two Techniques: Integrating $x/\sqrt[3]{ax+b}$

In this section, we will compare the difficulty and complexity of two different substitution techniques in evaluating the following symbolic integral:

$$I = \int \frac{x}{\sqrt[3]{ax+b}} dx$$

Method 1: Power Substitution

Using power substitution, we aim to do away with the cube root in the denominator by choosing an algebraic substitution which will remove it from our consideration by means of the laws of indices. The “standard” power substitution would be as follows:

$$u^3 = ax + b \text{ so that } \sqrt[3]{ax+b} = \sqrt[3]{u^3} = u$$

We now have to derive an expression for dx . In order to do this, let's rearrange our substitution formula in the following way:

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$$ax = u^3 - b \text{ so that } x = \frac{u^3 - b}{a} \text{ or } \frac{u^3}{a} - \frac{b}{a}$$

The second form of the above expression for x will make it easier to find an expression for dx by differentiation, as follows:

$$\frac{dx}{du} = \frac{3u^2}{a} \text{ so that } dx = \frac{3u^2}{a} du$$

We can combine all of these conclusions to write out our integral in terms of u as follows:

$$\int \frac{\left(\frac{u^3 - b}{a}\right)}{u} \cdot \frac{3u^2}{a} du = \int \frac{u^3 - b}{au} \cdot \frac{3u^2}{a} du = \int \frac{3u^5 - 3bu^2}{a^2 u} du = \frac{3}{a^2} \int (u^4 - bu) du$$

Despite requiring some careful algebra to get there, the integral in the final right-hand side is quite easily evaluated as follows:

$$\frac{1}{5}u^5 - \frac{b}{2}u^2 + C$$

And once we reapply the coefficient which we took outside the integral sign, we obtain the following solution in terms of u :

$$\frac{3}{5a^2}u^5 - \frac{3b}{2a^2}u^2 + C$$

In order to take the final step of reversing the substitution, we now recall that from our original substitution formula, we were able to state that:

$$u = \sqrt[3]{ax + b} = (ax + b)^{1/3}$$

We can then give the following final solution in terms of x :

$$I = \frac{3}{5a^2}(ax + b)^{5/3} - \frac{3b}{2a^2}(ax + b)^{2/3} + C$$

Method 2: The Substitution $u = ax + b$

In [Part 1 Section 8](#) (and, more generally, in [Section 3](#) above) we noted that it was possible to evaluate integrals of the form:

$$\int (Ax + B)(Cx + D)^p dx$$

using the substitution $u = Cx + D$. (Note that I have capitalised the coefficients here, since our current integrand already contains the lower-case coefficients a and b .) The integral we are considering here has exactly this form, where:

$$A = 1, B = 0, C = a, D = b \text{ and } p = \frac{1}{3}$$

We will therefore evaluate the integral this time by making the substitution:

$$u = ax + b \text{ so that } ax = u - b \text{ and } x = \frac{u - b}{a}$$

Differentiating without further ado this time, we find that:

$$\frac{du}{dx} = a \text{ so that } dx = \frac{1}{a} du$$

It's always good to have an expression for dx which is only a constant multiple of du , so let's now put all this into our original integral in order to obtain an integral in terms of u . We have:

$$\int \frac{\left(\frac{u-b}{a}\right)}{u^{1/3}} \cdot \frac{1}{a} du = \int \frac{u-b}{au^{1/3}} \cdot \frac{1}{a} du = \int \frac{u-b}{a^2 u^{1/3}} du = \frac{1}{a^2} \int (u^{2/3} - bu^{-1/3}) du$$

The final right-hand side here is obtained using the laws of indices on the following expansion of the integrand after the coefficient $1/a^2$ has been moved outside the integral sign:

$$\frac{u-b}{u^{1/3}} = \frac{u}{u^{1/3}} - \frac{b}{u^{1/3}}$$

The integral it contains can be evaluated using the standard rule for the integration of simple powers to give the following result:

$$\frac{3}{5} u^{5/3} - \frac{3b}{2} u^{2/3} + C$$

Restoring the coefficient $1/a^2$ gives the following overall solution in terms of u :

$$\frac{3}{5a^2} u^{5/3} - \frac{3b}{2a^2} u^{2/3} + C$$

If we look back at the solution in x which we obtained by the first method, we will see that its pattern is already present in the above solution in terms of u . All we have to do is to reverse the simple substitution we made with this second method to arrive at the same solution as we did a few paragraphs ago, namely:

$$I = \frac{3}{5a^2} (ax + b)^{5/3} - \frac{3b}{2a^2} (ax + b)^{2/3} + C$$

Is One Method Better Than the Other?

On the face of it, neither of these two methods seems to have a clear advantage over the other. Both required some careful algebra to reduce the integral in terms of u to its final form, mainly since in both cases the expression for x obtained by rearranging the substitution formula produced a rational expression in the numerator of the integrand.

The power substitution method had its usual effect of giving us some high powers of u to integrate, but this should not cause us any real sweat. The only thing we might have wished to avoid, perhaps, would have been the integration of the fractional powers of u which appeared in the integral in u resulting from the second method. This is, however, very much a matter of taste, since if we keep our heads, or write out the working at each stage, the standard rule for the integration of powers works perfectly well. We might finally remark that the second method gave a particularly convenient expression for dx , and the reversal of the substitution was also particularly straightforward.

6 Integration of $\sec x$ Using Substitution

In [Part 1 Section 11](#), in the course of my remarks following [Example 1](#), I briefly mentioned the fact that the integral:

$$I = \int \sec x \, dx$$

could be evaluated using substitution, with the aid of some algebra. I will set out the method which is required here.

Once we actually know what the integral of $\sec x$ is, we may be able to see more clearly why we take the approach I am about to set out. Without assuming that knowledge, however, it is hard to see how we would stumble upon the particular trick which is called for.

First, bear in mind that we intend to use substitution to evaluate this integral. Since it's hard to see how to find a substitution which will work for the apparently simple integral as it stands, we might expect to have to modify the integrand by applying to it a factor which, while not altering its actual algebraic value, would open up the possibility of a substitution. We also have to bear in mind that the step in our algorithm for integration by substitution where we have to find an expression for dx before making the substitution might also provide us, as it has before, with the ability to do some cancellation or otherwise simplify the transformed integrand.

What I am about to do is simultaneously to multiply and divide our integrand by another expression in x . This will be tantamount to multiplying the integrand by 1, so its value will not be altered. However, it will insert some new material into the integral which will allow us to choose a substitution and take advantage of the corresponding expression for dx .

We will rewrite our integral as follows:

$$I = \int \sec x \, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx$$

Having done this, we will take the obvious next step and decide to try a substitution of $u = \sec x + \tan x$. We do this because it is a moderately simple compound expression which might provide a useful expression for dx . (While this is not really very convincing, it's about as much justification as I can give at this stage; I do give some indication as to how we might make this choice using a deductive approach rather than simple trial and error at the end of this section.)

Using our chosen substitution, we can deduce that:

$$\frac{du}{dx} = \sec x \tan x + \sec^2 x = \sec x (\sec x + \tan x)$$

using the standard derivatives for $\sec x$ and $\tan x$ which we quoted in the text and then factorising out $\sec x$. Pausing for just a moment to admire the pleasant almost-symmetric behaviour of our substitution expression under differentiation, we pass straight on to conclude that:

$$dx = \frac{1}{\sec x (\sec x + \tan x)} du = \frac{1}{u \sec x} du$$

remembering that we have chosen $u = \sec x + \tan x$.

Making these substitutions into our modified integral, it transforms into:

$$\int \sec x \cdot \frac{u}{u} \cdot \frac{1}{u \sec x} du = \int \frac{u \sec x}{u^2 \sec x} du = \int \frac{1}{u} du$$

The last step here follows from making all the available cancellations in the integrand.

So we see that, as if by conjuration, we have turned our integral into a standard integral in u , which integrates to give:

$$\ln|u| + C$$

Reversing the substitution, we are left with the solution we quoted in the text as a standard integral, namely:

$$I = \ln|\sec x + \tan x| + C$$

Having arrived here, we can now see why $u = \sec x + \tan x$ might suggest itself as a promising substitution. Of course, the initial “multiply-and-divide” step is needed to get us into a position to use this substitution, and the convenient shape of the substitution expression's derivative allows us to do some helpful cancellation.

We finally note that the integrand which results when the algebraic factor has been applied, as in the integral:

$$\int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx$$

is of the form $g'(f(x))f'(x)$, where:

$$g'(f(x)) = \frac{1}{\sec x + \tan x}, f(x) = \sec x + \tan x, f'(x) = \sec x (\sec x + \tan x)$$

and as such is in prime condition for the method of integration by substitution, using $u = f(x)$. (Note that the derivative of $f(x)$ is in this case actually a longer expression than $f(x)$ itself.) This shape of the integral after the multiplication step is another factor which might help to justify in retrospect the approach we have taken here.

Justifying the Method

Having seen that our integrand has been made to conform to a particular pattern, we can go a little further in justifying our choice of the algebraic expression we used as a factor. We would begin our line of reasoning by asking ourselves: is there an expression $A(x)$ which can be applied to our integrand in the following way:

$$I = \int \sec x \, dx = \int \sec x \frac{A(x)}{A(x)} \, dx$$

in order to get the integral into a form which can be approached using substitution?

Applying $A(x)$ in this way – ie, multiplying by it and then dividing by it – is tantamount to multiplying the original integrand by 1, and so this operation does not change its value.

However, the whole point of doing this is to change our integral into something of the following form:

$$\int g'(f(x))f'(x) \, dx$$

which, as we saw in [Part 1](#), is a form of integral which will respond easily to the method of substitution.

We can even be a little more prescriptive, and say (to generalise from the breakdown of the actual integrand immediately before this subsection) that one way in which this method might work for us is if we make the following assignments:

$$g'(f(x)) = \frac{1}{A(x)}, f(x) = A(x), f'(x) = A(x) \sec x$$

The last two of these statements allow us to connect the function $f(x)$ with its derivative as follows:

$$f'(x) = f(x) \sec x$$

If it makes the relationship we are looking for any clearer, we can express this in the form of a differential equation, setting $y = f(x)$, as follows:

$$\frac{dy}{dx} = y \sec x$$

where we want to find y as a function of x .

While I do not propose to set out a formal solution to this differential equation here, it is worth pointing out that one traditional method of solving it would require us to integrate $\sec x$, which is of course the thing we are already trying to do! (To use a favourite phrase of one of my university tutors, this “*begs the question*”.)

However, we can make some progress by considering some specific cases. The differential equation tells us that we are looking for a function of x whose derivative is found by multiplying the original function by $\sec x$. Well, we do know of two functions whose derivatives contain a factor of $\sec x$: these are as follows:

$$\frac{d}{dx} \tan x = \sec^2 x \text{ and } \frac{d}{dx} \sec x = \sec x \tan x$$

If we substitute each of these functions and their derivatives into our differential equation, however, we find to our disappointment that neither of them on their own will fit the requirements. However, we observe that both these derivatives have a common factor, namely $\sec x$; if we take out this common factor, we can additionally observe that differentiating $\tan x$ generates an extra factor of $\sec x$, and differentiating $\sec x$ generates an extra factor of $\tan x$.

This should tip us off to the possibility that something might be gained by adding the two trigonometric functions together – this gives us $\sec x + \tan x$, whose derivative is:

$$\sec x \tan x + \sec^2 x = \sec x (\tan x + \sec x)$$

We can immediately see that the bracketed expression on the right-hand side is simply $\sec x + \tan x$ in reverse order! So, if we just substitute:

$$y = \sec x + \tan x$$

into our differential equation, we find to our immense joy that it is indeed a solution.

The differential equation we created was simply a restatement of an earlier equation connecting $f(x)$ with its derivative, so that $y = f(x)$. We also stated a few lines higher up that we would like to be able to say that $f(x) = A(x)$, our mystery expression. We can therefore finally conclude that:

$$A(x) = \sec x + \tan x$$

and the way is now clear for us to apply a factor of $A(x)/A(x)$ to our original integrand of $\sec x$, happy in the knowledge that the method of substitution will lead us straight to a solution.

7 Integration of $x(3x + 5)^2$ Using Substitution

In [Part 2 Section 4 Example 3](#), we wanted to evaluate the following integral:

$$I = \int x(3x + 5)^2 dx$$

In the text, I wanted to illustrate how integration by parts could be applied to this kind of integral, while at the same time noting that the integrand was of precisely the form examined in detail in [Part 1 Section 8](#), where we applied algebraic substitutions.

Using the approach explained in that section, we would expect to solve this integral using the substitution $u = 3x + 5$. As discussed in the same section, this substitution gives us the simple relationship:

$$dx = \frac{1}{3} du$$

The first factor in our integrand is simply x ; simple algebraic rearrangement to make x the subject of our substitution expression gives us:

$$x = \frac{1}{3}(u - 5)$$

Putting all these facts into the original integral, we arrive at the following version in terms of u :

$$\int \frac{1}{3} \cdot \frac{1}{3} (u - 5) u^2 du = \frac{1}{9} \int (u^3 - 5u^2) du$$

This is a nice straightforward polynomial expression in u , which integrates term by term to yield the following solution:

$$\frac{1}{9} \left(\frac{1}{4} u^4 - \frac{5}{3} u^3 \right) + C = \frac{1}{9} u^3 \left(\frac{1}{4} u - \frac{5}{3} \right) + C$$

When we take the final step of reversing the substitution, this turns into:

$$\frac{1}{9} (3x + 5)^3 \left(\frac{1}{4} (3x + 5) - \frac{5}{3} \right) + C$$

The final factor inside the large brackets will simplify as follows:

$$\frac{1}{4} (3x + 5) - \frac{5}{3} = \frac{3}{4} x + \frac{5}{4} - \frac{5}{3} = \frac{3}{4} x - \frac{5}{12} = \frac{1}{4} \left(3x - \frac{5}{3} \right)$$

so that the solution now boils down to the rather pretty form:

$$\frac{1}{36} (3x + 5)^3 \left(3x - \frac{5}{3} \right) + C$$

It seems clear that, apart from the algebra we performed at the end in order to tidy up the solution in x , this was a lot less laborious than the work of evaluating the integral by parts which is set out in the text.

Does Each Technique Give the Same Solution?

Of course, the solution we have obtained by substitution seems at first sight to be nothing like the solution obtained by parts in the text. It will be instructive to confirm that both these solutions eventually reduce to the same thing.

In order to do this, let's first establish a useful control by multiplying out the integrand into a polynomial in x , and then integrating term by term. This should provide us with a solution in polynomial form; we can then compare the solutions provided by our two techniques with this by multiplying them out in turn into polynomial form.

A: Term-by-Term Integration. First, a reminder of the integral we have been examining:

$$I = \int x(3x + 5)^2 dx$$

In order to reduce this to a simple polynomial, we must first expand out the squared factor in brackets. This results in:

$$\int x(9x^2 + 30x + 25)dx$$

Next, we multiply out this expression to transform the integrand into a simple polynomial, as follows:

$$\int (9x^3 + 30x^2 + 25x)dx$$

We can integrate this term by term to obtain the following result:

$$\frac{9}{4}x^4 + 10x^3 + \frac{25}{2}x^2 + C$$

We won't factorise this at all, since we have already decided that an unvarnished polynomial form will make it easier to compare our results.

(I might remark here and now that, were this book not intended to explore various special techniques for evaluating integrals, this would certainly be my chosen method for dealing with this particular example!)

B: Integration by Parts. A look back at [Part 2 Section 4 Example 3](#) will confirm that immediately before we took our final factorisation step, integration by parts yielded the solution:

$$\frac{9}{4}x^4 + 10x^3 + \frac{25}{2}x^2 + C$$

which is identical to the solution we just obtained by taking a term-by-term approach. So we can feel fairly confident that integration by parts has given us the correct result.

C: Integration by Substitution. The result we obtained above in presenting a solution using substitution was as follows:

$$\frac{1}{36}(3x+5)^3\left(3x-\frac{5}{3}\right)+C$$

This is clearly very different in appearance from the polynomial solutions we obtained using the two methods just described. However, it is presented in a factorised form, and it is not impossible that this result could be similar to the result we would obtain if we were prepared to factorise the quartic polynomial which was produced by our two previously mentioned techniques.

(We might also remark that integration by substitution involves the reversal of a substitution as its final step. If our solution in terms of u is factorised as far as is practical, and our substitution was of the form $u = ax + b$, then we might well expect the final result to be a product of linear factors, as it is in this case.)

In order to check that this factorised solution corresponds to the polynomial solution we have seen above, we will need to expand it out as fully as possible. We will start by expanding out the cubed factor in brackets. This will be done using the Binomial Theorem, which tells us that:

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

Applying this formula to the case in point, we obtain:

$$(3x+5)^3 = 27x^3 + 135x^2 + 225x + 125$$

We will now multiply this cubic successively by each of the two terms in the right-hand factor, obtaining the following:

$$81x^4 + 405x^3 + 675x^2 + 375x - 45x^3 - 225x^2 - 375x - \frac{625}{3}$$

Gathering together like terms, we will arrive at this:

$$81x^4 + 360x^3 + 450x^2 - \frac{625}{3}$$

We must finally multiply through the fractional coefficient at the beginning of the expression, to arrive at long last at this result:

$$\frac{9}{4}x^4 + 10x^3 + \frac{25}{2}x^2 - \frac{625}{108} + C$$

(I have reinstated the constant of integration C at the final stage.) We can see (perhaps a little to our relief) that this is in fact the same solution as we obtained with our two other approaches, with the minor difference of an extra constant term. Indeed, we should probably not be surprised at the presence of a constant term at the end of a process involving the multiplication of a number of linear factors.

It is important, however, in confirming that this last solution is in fact as good a solution as the earlier two, to make a number of remarks about this additional constant and its implications.

We first take a brief look at what happens in the event of our having to evaluate a definite integral of the kind we have just dealt with. If we initially say that:

$$F(x) = \frac{9}{4}x^4 + 10x^3 + \frac{25}{2}x^2$$

that is, the solution to the indefinite integral without the additional constant of integration C and without the additional constant term which emerged from our substitution solution, we will all, no doubt, agree that we can say the following:

$$\int_a^b x(3x + 5)^2 dx = (F(b) + C) - (F(a) + C)$$

This is nothing more than a reiteration of the result of evaluating a definite integral. We note that the two instances of the constant C cancel each other out.

Furthermore, there is surely nothing to stop us also asserting the following:

$$\int_a^b x(3x + 5)^2 dx = \left(F(b) - \frac{625}{108} + C\right) - \left(F(a) - \frac{625}{108} + C\right)$$

since here the additional improper fraction terms will also cancel each other out.

We can therefore see that the insertion of an additional constant term to the solution of the indefinite integral makes no difference to the result we obtain when evaluating a definite integral.

A further remark would be the following: if we make the assignment:

$$C' = C - \frac{625}{108}$$

then we can rewrite the solution which the method of substitution gave as:

$$\frac{9}{4}x^4 + 10x^3 + \frac{25}{2}x^2 + C'$$

which is to say simply “the polynomial in x plus some constant” – and this is the usual way in which we interpret solutions to indefinite integrals.

This also means that if we are solving a problem where some initial conditions are given which allow us to provide a solution containing a *particular* constant term, then a solution obtained using methods A or B above will be equivalent to the factorised solution obtained by method C as long as we subtract our rational term from the final constant, according to the expression we gave above for C' .

A final remark: if we had the time and determination to factorise fully the polynomial solution into the product of a constant and some linear terms, the decomposition of the substitution solution which we have just undertaken should show us that it would be necessary to introduce an explicit constant term in order to achieve this factorisation. This explicit constant term could be legitimately “borrowed” from the constant of integration C by replacing it with the adjusted constant C' . The relationship between C and C' would be the one given in the expression for C' quoted above.

8 Integration of $x^2 / (x^2 + 1)$ by tangent-substitution

In a note at the end of [Part 2 Section 4 Example 4](#) I mentioned that the following integral:

$$I = \int \frac{x^2}{1 + x^2} dx$$

could be evaluated using tangent-substitution. I will carry out this process here as a useful reference, also to provide a comparison of different techniques.

Under tangent-substitution, we use the “reversed” substitution $x = \tan u$. This gives us, on differentiation:

$$\frac{dx}{du} = \sec^2 u, \text{ so that } dx = \sec^2 u \, du$$

We now put the substitution and the formula for dx into our original integral, and we find it transformed as follows:

$$\int \frac{\tan^2 u \sec^2 u}{1 + \tan^2 u} du$$

This can be simplified by noting that the trigonometric formula connecting $\tan x$ and $\sec x$ can be rearranged as follows:

$$\sec^2 x = \tan^2 x + 1$$

This allows us to substitute a much simpler version of the denominator, so that we arrive at:

$$\int \frac{\tan^2 u \sec^2 u}{\sec^2 u} du = \int \tan^2 u \, du$$

(The expression on the right-hand side is obtained by cancelling out the instances of $\sec^2 u$ in the numerator and denominator.)

Unless we are prepared to quote the standard result for the integral of $\tan^2 x$ which is mentioned in [Appendix 4 Section 2](#), we need to do something to make this new integrand easier to handle in order to be able to continue. Fortunately, the trigonometric formula we have already used once will again ride to our rescue. This time, we can quote:

$$\tan^2 x = \sec^2 x - 1$$

which is another rearrangement of the formula connecting $\tan x$ and $\sec x$. This lets us rewrite our integral in u in the following form:

$$\int (\sec^2 u - 1) du$$

At this point, we should immediately be able to recognize $\sec^2 u$ as the derivative of $\tan u$, so that this integrates to:

$$\tan u - u + C$$

Reversing the substitution, and remembering that our substitution formula can be rearranged to give $u = \tan^{-1} x$, we have the following final solution:

$$I = x - \tan^{-1} x + C$$

which is the same result we obtained using the simple algebraic conversion of our integrand in [Part 2](#).

9 Integrating Functions of the Form $x^n / (x^2 + 1)$: An Investigation

In the notes at the end of [Part 2 Section 7](#), I mentioned the usefulness of being able to integrate rational functions of the form:

$$\frac{x^n}{x^2 + 1}$$

These were seen to be particularly helpful when integrating products of functions where one of the factors is a logarithmic or inverse trigonometric function.

In [Appendix 4 Section 2](#), I have set out as “standard” integrals the instances where n is 1, 2 or 3, noting where in the text we have seen how to carry out the integration. What I want to do here is to take a quick look at what happens if we consider values of n greater than 3.

The Case Where $n = 4$

Let's look first at the integral:

$$I = \int \frac{x^4}{x^2 + 1} dx$$

If we start by rewriting this as follows:

$$I = \int x \cdot \frac{x^3}{x^2 + 1} dx$$

we will have an integrand which comprises a product of x and something we already know how to integrate, namely the case of our general rational expression where $n = 3$.

We should therefore proceed by parts, choosing them so that we end up integrating the more heavyweight rational function whose integral we already know. We therefore make the following choices:

$$u = x \text{ and } \frac{dv}{dx} = \frac{x^3}{x^2 + 1}$$

These choices have the following consequences:

$$\frac{du}{dx} = 1 \text{ and } v = \int \frac{x^3}{x^2 + 1} dx = \frac{1}{2}(x^2 - \ln(x^2 + 1))$$

Once we have set these down, the integration by parts formula gives us:

$$I = \int \frac{x^4}{x^2 + 1} dx = \frac{1}{2}x(x^2 - \ln(x^2 + 1)) - \frac{1}{2} \int (x^2 - \ln(x^2 + 1)) dx$$

To carry on the process, we will set:

$$I_1 = \int (x^2 - \ln(x^2 + 1)) dx = \frac{1}{3}x^3 - \int \ln(x^2 + 1) dx$$

Moving on one more step, we will now set:

$$I_2 = \int \ln(x^2 + 1) dx$$

Integrating by parts, by first considering the logarithmic function as multiplied by a factor of 1, we make the following choices as suggested by the **LIATE** rule:

$$u = \ln(x^2 + 1) \text{ and } \frac{dv}{dx} = 1$$

These have the following consequences (using the Chain Rule for differentiation):

$$\frac{du}{dx} = \frac{2x}{x^2 + 1} \text{ and } v = x$$

The integration by parts formula then gives us the following result for I_2 :

$$I_2 = \int \ln(x^2 + 1) dx = x \ln(x^2 + 1) - 2 \int \frac{x^2}{x^2 + 1} dx$$

At this point, we can justifiably feel that we are almost finished, since the integral on the right-hand side here is one we already know, namely the integral of a rational function of the pattern we are considering for the case $n = 2$. By reference to [Appendix 4 Section 2](#), we are able to write down:

$$\int \frac{x^2}{x^2 + 1} dx = x - \tan^{-1} x + C$$

We can now start putting all our partial results together in order to obtain a final expression for the integral I . Proceeding backwards step by step, we have the following (multiplying out where we can at each stage, so as to be able to gather together like terms at the end):

$$I_2 = x \ln(x^2 + 1) - 2(x - \tan^{-1} x) + C = x \ln(x^2 + 1) - 2x + 2 \tan^{-1} x + C$$

$$I_1 = \frac{1}{3}x^3 - I_2 = \frac{1}{3}x^3 - x \ln(x^2 + 1) + 2x - 2 \tan^{-1} x + C$$

and finally:

$$I = \frac{1}{2}x(x^2 - \ln(x^2 + 1)) - \frac{1}{2}I_1 = \frac{1}{2}x^3 - \frac{1}{2}x \ln(x^2 + 1) - \frac{1}{2}I_1$$

Substituting in the expression for I_1 will give us:

$$I = \frac{1}{2}x^3 - \frac{1}{2}x \ln(x^2 + 1) - \frac{1}{6}x^3 + \frac{1}{2}x \ln(x^2 + 1) - x + \tan^{-1} x + C$$

We can gather together some like terms in this long expression to neaten up our result. This will bring us to the following:

$$I = \frac{1}{3}x^3 - x + \tan^{-1} x + C$$

Now that we have completed that multi-stage integral, we are in a position to set out a table of the integrals of our template function for values of n from 1 to 4. Taking into account the “standard” integrals we have stated in [Appendix 4 Section 2](#), the table is as follows:

Value of n	Integral
1	$\frac{1}{2} \ln(x^2 + 1) + C$
2	$x - \tan^{-1} x + C$
3	$\frac{1}{2}x^2 - \frac{1}{2} \ln(x^2 + 1) + C$
4	$\frac{1}{3}x^3 - x + \tan^{-1} x + C$

We can make some initial, and possibly suggestive, observations based on this fairly small collection of data. If we adopt the following useful shorthand notation for the integral of our template rational function with a particular power of x in the denominator:

$$J_n = \int \frac{x^n}{x^2 + 1} dx$$

then we will see from our table that we can describe the form taken by the integrals where $n = 3$ and 4 as follows:

$$J_3 = P(x) - J_1$$

and

$$J_4 = P(x) - J_2$$

where $P(x)$ is, in each case, some polynomial in x . (I have used the letter J with a subscript here to avoid confusion with the use of a subscripted I to represent the various sub-problems of an integration problem requiring solution in several stages.)

This might suggest to us that we could expect to find a logarithmic function in the results of integration of our rational function for odd values of n , and an inverse tangent in the results for even values of n , based on our results for J_1 and J_2 . However, we will need two things if this very vague conjecture is to be borne out – we will need further results for higher values of n to provide us with more experimental data, so to speak; and we will then need to provide a formal proof for the validity of any formula we surmise may apply from consideration of these results.

A General Formula for J_n

It will help in our further investigation of the results of integrating these rational functions if we can establish some formula which allows us to represent what happens when we integrate the rational function with x^n in the numerator. We can attempt to produce such a formula by generalizing from the approach we took above in integrating the case where $n = 4$.

Using the same approach, then, for the general case, we can write:

$$J_n = \int \frac{x^n}{x^2 + 1} dx = \int x \cdot \frac{x^{n-1}}{x^2 + 1} dx$$

If we are prepared to gloss over for the moment any difficulties involved, and assume that we have some useful way of integrating J_{n-1} , we can then proceed by parts, making the following choices:

$$u = x \text{ and } \frac{dv}{dx} = \frac{x^{n-1}}{x^2 + 1}$$

These choices lead to the following consequences:

$$\frac{du}{dx} = 1 \text{ and } v = \int \frac{x^{n-1}}{x^2 + 1} dx = J_{n-1}$$

The integration by parts formula now gives us the following:

$$J_n = \int \frac{x^n}{x^2 + 1} dx = xJ_{n-1} - \int J_{n-1} dx$$

This is all very well as far as it goes, but it does leave us with one concern: what happens if we find that we can't integrate J_{n-1} ? Since this is an open question for the present, we can't really claim that this represents any kind of reduction formula for integrals of the type we are considering.

Once again, we are still left in need of more data with which to work.

Two Useful Abbreviations

Before we proceed to integrate examples of our template function for higher values of n than 4, we will find it useful to abbreviate two of the terms which we saw appearing repeatedly in the table of integrals for the first few values of n which we were able to set out above.

We will make the following abbreviations:

$$L(x) = \frac{1}{2} \ln(x^2 + 1) \text{ and } T(x) = \tan^{-1} x$$

We can see that $L(x)$ appears in the integrals where $n = 1$ or 2; also, $T(x)$ appears where $n = 2$ or 4. Since the general formula for J_n which we gave above requires us to integrate J_{n-1} , it will be important to remind ourselves of what happens when we integrate $L(x)$ and $T(x)$.

Fortunately, we have done almost all the necessary work to be able to set these integrals out without further effort. A look back at the work we did earlier in this section for the case $n = 4$ will remind us that in the working we carried out, the intermediate integral I_2 was in fact the integral of $2L(x)$. We can therefore make the necessary minor amendment to the final value of I_2 to take into account the extra coefficient, and obtain the following:

$$\int L(x) dx = xL(x) - x + T(x) + C$$

As regards the integral of $T(x)$, this was obtained in [Part 2 Section 8 Example 2](#) using integration by parts and treating the inverse tangent function as being multiplied by 1; we obtained the following result:

$$\int T(x) dx = xT(x) - L(x) + C$$

From these two integrals, we can see that $L(x)$ and $T(x)$ exhibit a partial symmetry under integration. Combined with the general formula for J_n which we were able to obtain using the integration by parts formula as a starting point, this behaviour offers some explanation of

the alternation between $L(x)$ and $T(x)$ in the table of integrals we set out above. This will also be useful to us in deriving a general description of the behaviour of our rational function for any given value of n .

The Remaining Cases Where $n \leq 8$

I will now proceed to evaluate the integrals of our rational function where n is made to take the next few values, from 5 to 8. This will allow us to present a more extensive version of the table of integrals we displayed earlier, and from that, we might hope to proceed to some speculation about a general formula for these integral, whatever the value of n .

I will use the formula for J_n in terms of J_{n-1} which was set out above to derive each integral in the sequence, with J_4 , which we derived from first principles earlier in this section, as a starting point. Each case will be covered briefly, using the abbreviations $L(x)$ and $T(x)$ which we defined above.

First, the case $n = 5$. Our formula gives us:

$$J_5 = xJ_4 - \int J_4 dx$$

The right-hand side expands as follows:

$$\frac{1}{3}x^4 - x^2 + xT(x) - \frac{1}{12}x^4 + \frac{1}{2}x^2 - \int T(x)dx$$

and the integral formula we stated above gives us this:

$$\frac{1}{3}x^4 - x^2 + xT(x) - \frac{1}{12}x^4 + \frac{1}{2}x^2 - xT(x) + L(x) + C$$

Gathering together like terms and taking account of terms which cancel each other out, we end up with:

$$\frac{1}{4}x^4 - \frac{1}{2}x^2 + L(x) + C$$

Next, the case $n = 6$. By our formula:

$$J_6 = xJ_5 - \int J_5 dx$$

Expanding the right-hand side gives:

$$\frac{1}{4}x^5 - \frac{1}{2}x^3 + xL(x) - \frac{1}{20}x^5 + \frac{1}{6}x^3 - \int L(x)dx$$

By our integral formula, this becomes:

$$\frac{1}{4}x^5 - \frac{1}{2}x^3 + xL(x) - \frac{1}{20}x^5 + \frac{1}{6}x^3 - xL(x) + x - T(x) + C$$

and this reduces to the following final result:

$$\frac{1}{5}x^5 - \frac{1}{3}x^3 + x - T(x) + C$$

Next, the case $n = 7$. By our formula:

$$J_7 = xJ_6 - \int J_6 dx$$

Expanding the right-hand side gives:

$$\frac{1}{5}x^6 - \frac{1}{3}x^4 + x^2 - xT(x) - \frac{1}{30}x^6 + \frac{1}{12}x^4 - \frac{1}{2}x^2 + \int T(x)dx$$

By our integral formula, this becomes:

$$\frac{1}{5}x^6 - \frac{1}{3}x^4 + x^2 - xT(x) - \frac{1}{30}x^6 + \frac{1}{12}x^4 - \frac{1}{2}x^2 + xT(x) - L(x) + C$$

and this reduces to the following final result:

$$\frac{1}{6}x^6 - \frac{1}{4}x^4 + \frac{1}{2}x^2 - L(x) + C$$

Finally, the case $n = 8$. By our formula:

$$J_8 = xJ_7 - \int J_7 dx$$

Expanding the right-hand side gives:

$$\frac{1}{6}x^7 - \frac{1}{4}x^5 + \frac{1}{2}x^3 - xL(x) - \frac{1}{42}x^7 + \frac{1}{20}x^5 - \frac{1}{6}x^3 + \int L(x)dx$$

By our integral formula, this becomes:

$$\frac{1}{6}x^7 - \frac{1}{4}x^5 + \frac{1}{2}x^3 - xL(x) - \frac{1}{42}x^7 + \frac{1}{20}x^5 - \frac{1}{6}x^3 + xL(x) - x + T(x) + C$$

and this reduces to the following final result:

Supplementary Examples and Investigations

$$\frac{1}{7}x^7 - \frac{1}{5}x^5 + \frac{1}{3}x^3 - x + T(x) + C$$

We are now in a position to expand our earlier table of integrals to cover the first 8 natural number values of n . The table now looks as follows:

Value of n	Integral
1	$\frac{1}{2}\ln(x^2 + 1) + C$
2	$x - \tan^{-1}x + C$
3	$\frac{1}{2}x^2 - \frac{1}{2}\ln(x^2 + 1) + C$
4	$\frac{1}{3}x^3 - x + \tan^{-1}x + C$
5	$\frac{1}{4}x^4 - \frac{1}{2}x^2 + \frac{1}{2}\ln(x^2 + 1) + C$
6	$\frac{1}{5}x^5 - \frac{1}{3}x^3 + x - \tan^{-1}x + C$
7	$\frac{1}{6}x^6 - \frac{1}{4}x^4 + \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2 + 1) + C$
8	$\frac{1}{7}x^7 - \frac{1}{5}x^5 + \frac{1}{3}x^3 - x + \tan^{-1}x + C$

We can now record some more general comments on the behaviour of the integrals we are considering for this extended range of values of n .

First, we note that for odd values of n , the integral is of the form:

$$P(x) \pm L(x) + C$$

(In the case $n = 1$, $P(x)$ is the “trivial” polynomial, all of whose coefficients are zero, so that $P(x) = 0$.) The sign of $L(x)$ starts as a plus sign for $n = 1$ and cycles between plus and minus for $n = 3, 5$ and 7 .

For even values of n , the integral has the form:

$$P(x) \pm T(x) + C$$

starting at $n = 2$ with the polynomial $P(x) = x$. Here, the sign of $T(x)$ starts with a minus sign for $n = 2$, and cycles between minus and plus for $n = 4, 6$ and 8 .

We should additionally notice that for every value of n the order of the polynomial $P(x)$ is $n - 1$. For odd values of n , $P(x)$ contains only even powers of x ; and for even values of n , $P(x)$ contains only odd powers of x . Furthermore, every single term of every instance of $P(x)$ has the form:

$$\pm \frac{1}{k}x^k$$

And the sign of each term of $P(x)$ alternates between plus and minus, starting with a plus sign for the highest-order term at the beginning.

A Better General Formula for J_n

We can go further than these observations, and suggest a general formula for J_n , which will certainly be satisfied by all of the eight cases we have evaluated so far.

We might think that we will need separate formulae for the odd and even cases, since as we have noted above, the final term in the integral for the odd cases is $L(x)$, while the final term for the even cases is $T(x)$.

A careful examination of the table will verify that, for odd cases where $n \geq 3$, we find that:

$$J_n = \frac{1}{n-1} x^{n-1} - J_{n-2}$$

A further look back at the table will also confirm that (perhaps surprisingly) exactly the same relation holds between the cases for even values of n , at least where $n \geq 4$.

If we can accept the integrals for $n = 1$ and 2 as obtainable from direct integration – and we have in fact integrated these elsewhere in the text (the exact references are given in [Appendix 4 Section 2](#)) – then we seem to have here a useful reduction formula for the class of integrals we are investigating – and this formula does not involve the use of an integral sign, as our earlier one did.

A Proof of the Reduction Formula

It is surprisingly easy to provide a proof that this reduction formula holds for all values of $n \geq 3$, with the aid of a little simple calculus and algebra.

The key to getting started is to notice that the first term on the right-hand side can in fact be obtained by integrating x^{n-2} – in other words:

$$\int x^{n-2} dx = \frac{1}{n-1} x^{n-1} + C$$

This follows from the standard formula for the integral of a power of x .

Now, if we can show that:

$$\frac{x^n}{x^2 + 1} = x^{n-2} - \frac{x^{n-2}}{x^2 + 1}$$

(which at first seems a slightly unusual identity), we will be able to complete the proof very simply. The chain of equalities which achieves this is as follows:

$$x^{n-2} - \frac{x^{n-2}}{x^2 + 1} = x^{n-2} \frac{x^2 + 1}{x^2 + 1} - \frac{x^{n-2}}{x^2 + 1} = \frac{x^n + x^{n-2} - x^{n-2}}{x^2 + 1} = \frac{x^n}{x^2 + 1}$$

We managed this by multiplying the first term by an expression in x whose value is equal to 1 – so nothing about the value of the term is altered by that. Our choice of the factor used – in particular its denominator – allows us to rewrite our initial expression as one single fraction with a denominator of $x^2 + 1$. Finally, the x^{n-2} terms are seen to cancel one another out, and we are left with the equality we need.

The final step is to integrate both sides of the above equality, as follows:

$$\int \frac{x^n}{x^2 + 1} dx = \int \left(x^{n-2} - \frac{x^{n-2}}{x^2 + 1} \right) dx = \frac{1}{n-1} x^{n-1} - \int \frac{x^{n-2}}{x^2 + 1} dx$$

This, of course – using our abbreviations for the integrals we are investigating – is equivalent to the equality:

$$J_n = \frac{1}{n-1} x^{n-1} - J_{n-2}$$

and this is precisely the reduction formula we wanted to prove.

10 Integration of $e^{ax} \cos bx$ by Solving an Equation

In [Part 2 Section 12 Examples 1 and 2](#), we examined the integration of two functions comprising the product of an exponential function and a trigonometric function. These functions are of particular interest in the modelling of various physical systems, and so it will be instructive to demonstrate here the integration of a general case, in particular the following integral:

$$I = \int e^{ax} \cos bx \, dx$$

We dealt with the functions in the examples mentioned by applying integration by parts twice; this resulted in an equation for the value of I which we were able to solve by ordinary algebra. We might expect to follow a similar procedure in this general case.

To begin with, then, we will choose parts according to the **LIATE** rule as follows:

$$u = \cos bx \text{ and } \frac{dv}{dx} = e^{ax}$$

These choices have the following consequences:

$$\frac{du}{dx} = -b \sin bx \text{ and } v = \frac{1}{a} e^{ax}$$

and with these in mind, the integration by parts formula gives us:

$$\int e^{ax} \cos bx \, dx = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} \int e^{ax} \sin bx \, dx$$

where we have taken into account the presence of a minus sign in our expression for du/dx and moved the constant coefficient b/a outside the right-hand side integral sign.

We can move to the next stage by setting I_1 equal to the right-hand side integral, as follows:

$$I_1 = \int e^{ax} \sin bx \, dx$$

To solve this second integral, we will make the following choice of parts, as usual observing the **LIATE** rule:

$$u = \sin bx \text{ and } \frac{dv}{dx} = e^{ax}$$

These choices give us the following:

$$\frac{du}{dx} = b \cos bx \text{ and } v = \frac{1}{a} e^{ax}$$

Putting these into the integration by parts formula, we arrive at the following equation for I_1 :

$$I_1 = \int e^{ax} \sin bx \, dx = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} \int e^{ax} \cos bx \, dx$$

once again moving the coefficient b/a outside the right-hand side integral sign.

We should note that the original integral I has now reappeared on the right-hand side (as we might have expected based on our experience of the concrete cases from [Part 2](#)). Restating the above equation using the I -notation I have adopted, we have:

$$I_1 = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} I$$

Recalling that the original integration by parts formula for I can be restated as:

$$I = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} I_1$$

We can combine both of these equations to give the following more complete expression for I :

$$I = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} \left(\frac{1}{a} e^{ax} \sin bx - \frac{b}{a} I \right) = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} I$$

The simple algebraic rearrangement of adding $(b^2/a^2)I$ to both sides gives us this:

$$\left(1 + \frac{b^2}{a^2} \right) I = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx$$

To make the next piece of algebra easier, we should rewrite the coefficient of I on the left-hand side as follows:

$$1 + \frac{b^2}{a^2} = \frac{a^2}{a^2} + \frac{b^2}{a^2} = \frac{a^2 + b^2}{a^2}$$

It's now a simple matter to divide both sides by this expression to obtain the following final expression for I :

$$I = \frac{a}{a^2 + b^2} e^{ax} \cos bx + \frac{b}{a^2 + b^2} e^{ax} \sin bx + C$$

We have, of course, carried out some cancellation here with the existing coefficients of the trigonometric functions on the right-hand side; we must also now formally add the constant of integration C .

As a final step, we may prefer to factorise the expression on the right-hand side as follows:

$$I = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + C$$

This gives us a solution which is the product of an exponential function with the sum of two trigonometric functions. Various constants also make their presence felt – these are related to the constant coefficients of x which appear in the original integral.

In closing, it will do no harm to confirm our result by checking that it correctly predicts the solution we obtained by performing the integration in [Part2 Section 12 Example 2](#). The integral we dealt with there was as follows:

$$\int e^{2x} \cos 3x \, dx$$

This is a specific case of our general integral, where $a = 2$ and $b = 3$. Putting these values into our general solution, we find that it predicts the following:

$$\int e^{2x} \cos 3x \, dx = \frac{e^{2x}}{2^2 + 3^2} (2 \cos 3x + 3 \sin 3x) + C$$

The right-hand side is equal to

$$\frac{1}{13} e^{2x} (2 \cos 3x + 3 \sin 3x) + C$$

which is the solution we obtained by explicit working in [Part 2 Section 12](#). (We also have the added bonus of a clearer understanding of why the coefficient $1/13$ is present in this solution: the denominator 13 is the sum of the squares of the constant coefficients of x in the integrand - namely 2 and 3.)

11 Integrating Higher Powers of $\sec x$.

In [Example 3 of Part 2 Section 12](#), we integrated $\sec^3 x$ by applying integration by parts and obtaining an equation which we were able to solve in order to produce our solution. The first step of this procedure was to split the integrand into a product of two functions, as follows:

$$\sec^3 x = \sec x \cdot \sec^2 x$$

We were then able to recognize the second factor, $\sec^2 x$, as the derivative of $\tan x$, so that we could choose parts so that this factor was treated as dv/dx .

I remarked in my notes at the end of that example that the same technique – of exposing a factor of $\sec^2 x$ – would be useful when integrating higher powers of $\sec x$. I will exhibit this approach here, taking the general case of evaluating the integral:

$$I_n = \int \sec^n x dx$$

I have named the integral as I with a subscript of n since we will find that the result of our efforts will be to describe the result in terms of integrals of lower powers of $\sec x$ - in other words, we will obtain a reduction formula. In order to avoid difficult questions arising, we will also limit our consideration here to the range of values of n which are greater than or equal to 3.

The reasoning is moderately simple. As explained above, we will begin by rewriting the integrand as the product of a function with $\sec^2 x$, which we know to be the derivative of $\tan x$. This gives us:

$$I_n = \int \sec^n x dx = \int \sec^{n-2} x \cdot \sec^2 x dx$$

We will now choose parts, bearing in mind that we have already decided which part will take the role of dv/dx :

$$u = \sec^{n-2} x \text{ and } \frac{dv}{dx} = \sec^2 x$$

We then have the following consequences (again, we are fully expecting one of these, and the other is obtained using the Chain Rule):

$$\frac{du}{dx} = (n-2)\sec^{n-2} x \tan x \text{ and } v = \tan x$$

The integration by parts formula then gives us:

$$\int \sec^n x dx = \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x dx$$

Just as we did when dealing with the equivalent stage in the integration of $\sec^3 x$, we will use the trigonometric identity for $\tan^2 x$ (quoted in [Appendix 4 Section 3](#)) to rewrite the right-hand side integral as follows:

$$\int \sec^{n-2} x \tan^2 x dx = \int \sec^{n-2} x (\sec^2 x - 1) dx = \int \sec^n x dx - \int \sec^{n-2} x dx$$

We can now restate the result we obtained using the integration by parts formula as follows, making use of the subscripted I notation I introduced at the beginning of the discussion:

$$I_n = \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2}$$

Moving all the instances of I_n to the left-hand side, we have:

$$(n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}$$

which gives us our final reduction formula of:

$$I_n = \frac{1}{n-1} \sec^{n-2} x \tan x - \frac{n-2}{n-1} I_{n-2}$$

As a useful confirmation of the correctness of this result, we will recall that in the case where $n = 3$, which was examined in [Part 2 Section 12](#), our result could be written using our current notation as:

$$I_3 = \frac{1}{2} \sec x \tan x - \frac{1}{2} I_1$$

I_1 is, of course, nothing more than:

$$\int \sec x = \ln|\sec x + \tan x| + C$$

and so we see that our reduction formula has succeeded in predicting the result we obtained in [Part 2](#).

12 Integrating $\sec x \tan^2 x$

In [Part 2 Section 12 Example 3](#), we applied integration by parts to an unlikely candidate for this technique of integration, namely the function $\sec^3 x$. We were able to do this by expressing the integrand as the product of two functions, and noticing that one of these, $\sec^2 x$, was the derivative of the well-known function $\tan x$. We therefore chose this part of the product of two functions to be dv/dx , which allowed us to use the integration by parts formula to create an equation which was solved in order to produce the result we were looking for.

I remarked in this context that we might be able to get a similar advantage when integrating a product of functions where one of the factors was $\sec x \tan x$, which we know to be the derivative of $\sec x$.

We will look at the integration of such a function here. The integral we will investigate actually appeared in the course of the integration of $\sec^3 x$ (where we dealt with it using a trigonometric formula) and it is as follows:

$$I = \int \sec x \tan^2 x \, dx$$

This integral will respond to integration by parts, as we shall see. As in the case of $\sec^3 x$, we will need to state the integrand as a suitable product of two functions. It is, of course, already the product of $\sec x$ and $\tan^2 x$; but since the **LIATE** rule will not assist us when both the functions in our product are of the same type, and neither of these factors immediately claims our attention as a common derivative, we need to try another restatement of this integrand if we are to get anywhere very easily.

As suggested in the preceding observations, however, if we choose to rewrite the integrand as follows:

$$\sec x \tan^2 x = \sec x \tan x \cdot \tan x$$

we will recognise straight away that one of the factors in the restatement is $\sec x \tan x$, which is the derivative of $\sec x$. We can therefore choose parts as follows:

$$u = \tan x \text{ and } \frac{dv}{dx} = \sec x \tan x$$

The consequences of these choices are:

$$\frac{du}{dx} = \sec^2 x \text{ and } v = \sec x$$

and the integration by parts formula then leads us to the following:

$$\int \sec x \tan^2 x \, dx = \sec x \tan x - \int \sec^3 x \, dx$$

Fortunately, we have performed the integration of $\sec^3 x$ in [Part 2 Section 12 Example 3](#). Substituting the solution we obtained there into the above equation, we can therefore conclude that:

$$\int \sec x \tan^2 x \, dx = \sec x \tan x - \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln |\sec x + \tan x| + C$$

Gathering together the terms in $\sec x \tan x$ gives us the final solution:

$$I = \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln |\sec x + \tan x| + C$$

which we note with interest is almost the same as the integral of $\sec^3 x$, with the plus sign replaced by a minus sign.

13 A Formula for a Definite Integral of $\sin^n x$

In [Part 2 Section 13 Example 1](#), we obtained a quite elegant and simple reduction formula for the following definite integral:

$$J_n = \int_0^{2\pi} \sin^n x dx$$

The reduction formula we were able to derive was as follows:

$$J_n = \frac{n-1}{n} J_{n-2}$$

I will now conduct an investigation into the implications of this reduction formula, and suggest a direct formula for J_n which does not depend on another integral of the (J_k) sequence. I will then offer a proof of this new formula's correctness using mathematical induction.

The Investigation

We can see that if we apply the reduction formula repeatedly (and we did this for a specific value of n when we evaluated the indefinite integral of $\cos^7 x$ at the beginning of [Part 2 Section 13](#)), we will obtain the following sequence of equalities:

$$J_n = \frac{n-1}{n} J_{n-2} = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} J_{n-4} = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} J_{n-6} = \dots$$

Having noted the pattern which is being established in the above chain of equalities, and extending this as far as it will go, we would expect eventually to obtain:

$$J_n = \frac{(n-1)(n-3)(n-5) \dots 3.1}{n(n-2)(n-4) \dots 4.2} J_0$$

in the cases where n is even, and:

$$J_n = \frac{(n-1)(n-3)(n-5) \dots 4.2}{n(n-2)(n-4) \dots 5.3} J_1$$

where n is odd.

(Note that we must consider these two cases separately, since our reduction formula requires us to count down from n in steps of 2 – this will eventually bring us to 0 where n is even and 1 where n is odd.)

At this point, we must make an important observation about the value of J_1 which will render our remaining tasks a lot easier. That observation can be summarised as follows:

$$J_1 = \int_0^{2\pi} \sin x \, dx = \left[-\cos x \right]_0^{2\pi} = (-1) - (-1) = 0$$

(I might add that consideration of the graph of $y = \sin x$ would lead us more informally to the same conclusion: interpreting the definite integral as the area between the curve and the x -axis, we should be able to appreciate that the positive area between $x = 0$ and $x = \pi$ is cancelled out by the equal but negative area between $x = \pi$ and $x = 2\pi$.)

Looking back at our equation for J_n where n is odd, this fact about J_1 implies that J_n is actually equal to zero whenever n is odd! This means that from now on, we only need to think about J_n for *even* values of n : in other words, values of the definite integrals J_{2k} for natural number values of k .

One further useful observation is necessary (though somewhat less radical in its implications) – and that concerns the value of J_0 . Once more, we can summarise the reasoning very briefly as follows:

$$J_0 = \int_0^{2\pi} 1 \, dx = \left[x \right]_0^{2\pi} = 2\pi - 0 = 2\pi$$

We can therefore immediately rewrite our equation for J_n where n is even as follows:

$$J_n = \frac{(n-1)(n-3)(n-5) \dots 3.1}{n(n-2)(n-4) \dots 4.2} 2\pi$$

Can this be made any more succinct? Well, we might start by recalling that we are now only interested in J_n for even values of n – therefore, we should rewrite our equation as an expression for J_{2k} – we can confidently express n as $2k$ since we are now only considering even values of n , and any even natural number n can be written as $2k$ for some natural number k .

Making this substitution in our formula for J_n where n is even gives us the following new version:

$$J_{2k} = \frac{(2k-1)(2k-3)(2k-5) \dots 3.1}{2k(2k-2)(2k-4) \dots 4.2} 2\pi$$

Now, the denominator of the fraction in the right-hand side expression contains a factor of 2 within each of the factors in the version shown. We can extract these factors, and rewrite the denominator as follows:

$$2k(2k-2)(2k-4) \dots 4.2 = 2k \cdot 2(k-1) \cdot 2(k-2) \dots (2.2) \cdot (2.1)$$

In the right-hand side, we can now see k factors of 2; we can lump these together as one single factor of 2^k , after which we will be left with $k(k-1)(k-2) \dots 2.1$. This can be expressed using the factorial notation as $k!$ – so we can now rewrite the denominator of our fraction as follows:

$$2k(2k-2)(2k-4) \dots 4.2 = 2^k k!$$

What about the numerator? It looks like a rather harder nut to crack at first, until we notice that it would be equal to:

$$(2k)! = 2k(2k-1)(2k-2)(2k-3) \dots 4.3.2.1$$

were it not for the fact that the first factor and every second subsequent factor in the expansion of the factorial expression is missing. This is actually a useful realisation, since it means that we can rewrite our numerator as follows:

$$(2k-1)(2k-3)(2k-5) \dots 3.1 = \frac{(2k-1)(2k-2)(2k-3) \dots 4.3.2.1}{2k(2k-2)(2k-4) \dots 4.2}$$

The expression in the denominator here contains the factors $2k$, $(2k-2)$, $2k-4$ and so on – these all cancel out with the corresponding factors in the numerator – after this cancelling, we are only left with the factors shown on the left-hand side. Now, we have already established a little earlier that the artificial denominator we introduced above can be abbreviated by the expression $2^k k!$ - so we can therefore now rewrite our expression for J_{2k} as follows:

$$J_{2k} = \frac{(2k-1)(2k-3)(2k-5) \dots 3.1}{2k(2k-2)(2k-4) \dots 4.2} 2\pi = \frac{(2k)!}{2^k k! \cdot 2^k k!} 2\pi$$

The right-hand side expression can be considerably tidied up by combining the two factors of 2^k (using the laws of indices) and then cancelling out the factor of 2 which appears before π to obtain the following:

$$J_{2k} = \frac{(2k)!}{2^{2k-1} (k!)^2} \pi$$

So far, so good: we now have a formula for j_{2k} which does not depend on any other integral of the form J_r . Can we make it any neater? Well, we can if you are happy to allow me to pull a metaphorical rabbit out of a hat, and have recourse to the notation used to represent the binomial coefficients.

To get started down this road, let's first rearrange our formula a little so as to gather all the factorial terms together; this will make much clearer the usefulness of the binomial coefficient notation for our purposes. We will rewrite it as follows:

$$J_{2k} = \frac{(2k)!}{(k!)^2} \cdot \frac{\pi}{2^{2k-1}}$$

If you are familiar with this useful notation (which pops up again and again in the most unexpected mathematical places!), you will recall the following definition for the special notation used to identify these coefficients:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Whenever we are confronted with a fraction with factorial terms in both the numerator and denominator, it can be worthwhile to see if the fraction can be expressed using the binomial coefficient notation. In order to see if this is possible here, let's just equate the standard binomial coefficient form with the part of our formula which consists of factorial terms, and see if this will help us. We will try setting:

$$\frac{n!}{r!(n-r)!} = \frac{(2k)!}{(k!)^2}$$

and then investigate to see if there are values of n and r which will satisfy this equation. One pair of values which would work would be those obtained by simply equating the numerator and denominator on each side.

Taking the numerator first, we would have $n! = (2k)!$, which would be satisfied if $n = 2k$. Substituting this value for n into the left-hand side denominator, and simply equating the two denominators, we get:

$$r!(2k-r)! = (k!)^2$$

One value of r which will satisfy this would be obtained if we can satisfy both $r! = k!$ and $(2k-r)! = k!$. Fortunately, the natural implication of the first of these two conditions – $r = k$ – also satisfies the second condition: $(2k-k)! = k!$.

So this speculative approach brings us to the following useful conclusion: our fraction with two factorials in it *can* indeed be abbreviated using the binomial coefficient notation, as follows:

$$\binom{2k}{k} = \frac{(2k)!}{k!(2k-k)!} = \frac{(2k)!}{(k!)^2}$$

(This may be recognisable to readers with a thorough knowledge of the binomial coefficients and their properties as an easily derived –even “standard” – property of these coefficients.)

All this means that our formula for J_{2k} can now be written a little more simply as follows:

$$J_{2k} = \binom{2k}{k} \frac{\pi}{2^{2k-1}}$$

The Proof

This is certainly a very attractive result – perhaps the most elegant result anywhere in this book! However, the reasoning which led to it was a little sketchy and imprecise, so we must really feel obliged to treat the above discussion as no more than an investigation which suggested a suitable formula, and now take steps to provide a rigorous proof.

We will prove the validity of our formula using the method of mathematical induction. To summarise beforehand, we will first prove the formula for the case $k = 1$; next, we will *assume* that the formula is true when $k = r$ and then attempt to deduce from this that it also holds true when $k = r + 1$. The principle underlying proof by induction is that, having proved the truth of the formula for $k = 1$, this second step will entitle us to say that it must then also be true for $k = 2$. A similar justification can be given for any natural number value of k , by relying on the second inductive step to justify the truth of our formula each time the value of k is increased by 1, all the way up to the particular value we may be considering.

When carrying out these two steps, we will be able to rely on the reduction formula we obtained in [Part 2 Section 13 Example 1](#), which was as follows:

$$J_n = \frac{n-1}{n} J_{n-2}$$

So let's start by proving our explicit formula true when $k = 1$. In this case, our formula becomes:

$$J_2 = \binom{2}{1} \frac{\pi}{2} = \frac{2!}{(1!)^2} \cdot \frac{\pi}{2} = 2 \cdot \frac{\pi}{2} = \pi$$

Approaching the question by means of the reduction formula, we have the following:

$$J_2 = \frac{1}{2} J_0 = \frac{1}{2} \cdot 2\pi = \pi$$

(We established the value of J_0 as 2π earlier in this section, as part of the investigation which led to our formula in the first place.)

We have now shown that our explicit formula gives the same value for J_2 as is obtained from the reduction formula. So the first step of the proof by induction is complete.

We now move on to the second step, under which we will assume the truth of our formula for the case where $k = r$. This assumption can be written as follows:

$$J_{2r} = \binom{2r}{r} \frac{\pi}{2^{2r-1}} = \frac{(2r)!}{(r!)^2} \cdot \frac{\pi}{2^{2r-1}}$$

by simply substituting r for k . We now want to prove the formula for the case where $k = r + 1$. That is, we want to prove the following version of our formula:

$$J_{2(r+1)} = \binom{2r+2}{r+1} \frac{\pi}{2^{2r+1}} = \frac{(2r+2)!}{((r+1)!)^2} \cdot \frac{\pi}{2^{2r+1}}$$

(We have taken advantage in the above expressions of the facts that $2(r+1) = 2r+2$ and $2(r+1) - 1 = 2r+2 - 1 = 2r+1$ in order to neaten our notation.)

Looking back at our reduction formula, and making use of our assumption about the correctness of the formula for J_{2r} , we can obtain the following:

$$j_{2(r+1)} = \frac{2r+1}{2r+2} J_{2r} = \frac{2r+1}{2r+2} \cdot \frac{(2r)!}{(r!)^2} \cdot \frac{\pi}{2^{2r-1}} = \frac{(2r+1)!}{(r+1)! r!} \cdot \frac{\pi}{2^{2r}}$$

(Again, we have here used the identity $2(r+1) = 2r+2$; in the final step, we gathered factors of 2 in the denominator together to create $2 \cdot 2^{2r-1} = 2^{2r}$ using the laws of indices.)

Now, this does not quite provide the equivalence we are looking for. However, if we take the result of applying the reduction formula and multiply top and bottom by $2r+2$ – which is, of course, equal to $2(r+1)$ – we will obtain the following much more appropriate result:

$$j_{2(r+1)} = \frac{2r+2}{2(r+1)} \cdot \frac{(2r+1)!}{(r+1)! r!} \cdot \frac{\pi}{2^{2r}} = \frac{(2r+2)!}{2(r+1)! (r+1)!} \cdot \frac{\pi}{2^{2r}}$$

Just as we did a moment ago, we can gather together the factors of 2 in the denominator to obtain the following:

$$j_{2(r+1)} = \frac{(2r+2)!}{((r+1)!)^2} \cdot \frac{\pi}{2^{2r+1}} = \binom{2r+2}{r+1} \frac{\pi}{2^{2r+1}}$$

This finally shows that the reduction formula we obtained and the explicit formula we are trying to prove provide the same result for $j_{2(r+1)}$, based on the inductive assumption that this is true for J_{2r} . We have therefore completed the second step of our proof by induction, and thus proved that the formula we obtained in the course of our preceding investigation is correct.

14 A Formula for the Integral of $\ln^n x$

In [Part 2 Section 13 Example 3](#), I presented an extended discussion on reduction formulae for the indefinite integral:

$$I_n = \int \ln^n x \, dx$$

We derived two reduction formulae, one of which was simpler than the other. The simpler of the two formulae, which will form the basis of our discussion here, was as follows:

$$I_n = x \ln^n x - n I_{n-1}$$

In a similar manner to the previous section of this appendix, I will now investigate the prospects of deriving a direct formula for the general integral I_n which does not depend on any other integral of the (I_k) sequence, and I will then offer a proof of the formula by mathematical induction.

The Investigation

We'll start this investigation by setting up a table showing the solutions to the integrals I_n for the first few values of n starting with $n = 1$. This first solution is, of course, as follows:

$$I_1 = \int \ln x \, dx = x \ln x - x$$

a result we obtained in [Part 2 Section 8](#). We will be able to write down the next few entries in our table by making use of the reduction formula I have already quoted.

Here is the table of solutions for values of n from 1 to 6:

Integral	Solution
I_1	$x \ln x - x$
I_2	$x \ln^2 x - 2x \ln x + 2x$
I_3	$x \ln^3 x - 3x \ln^2 x + 6x \ln x - 6x$
I_4	$x \ln^4 x - 4x \ln^3 x + 12x \ln^2 x - 24x \ln x + 24x$
I_5	$x \ln^5 x - 5x \ln^4 x + 20x \ln^3 x - 60x \ln^2 x + 120x \ln x - 120x$
I_6	$x \ln^6 x - 6x \ln^5 x + 30x \ln^4 x - 120x \ln^3 x + 360x \ln^2 x - 720x \ln x + 720x$

We can immediately make a few simple observations. First, each of the integrals I_n comprises a series of terms of the form $kx \ln^i x$ where i ranges from n down to 1; the first of these terms always has a coefficient of 1. Second, at the end of this sequence we always find a term of the form kx (which could, of course, be regarded as a term of the form $kx \ln^0 x$) which has the same coefficient as the immediately preceding term of the form $kx \ln x$. Finally, the signs separating each of the terms in the series always start with a minus sign, then alternate regularly between plus and minus until the end of the series is reached.

We can give some explanation for this behaviour by examining what happens when we start with an expression for I_n using the reduction formula quoted above, and then expand it progressively by using versions of the reduction formula for I_{n-1} , I_{n-2} and so on. In order to do this, we recall the reduction formula for I_n :

$$I_n = x \ln^n x - n I_{n-1}$$

We can use this formula with some simple substitution to express the following indexed integrals:

$$I_{n-1} = x \ln^{n-1} x - (n-1) I_{n-2}$$

$$I_{n-2} = x \ln^{n-2} x - (n-2) I_{n-3}$$

Expanding I_{n-1} as it appears in the reduction formula for I_n , we have:

$$I_n = x \ln^n x - n(x \ln^{n-1} x - (n-1) I_{n-2}) = x \ln^n x - nx \ln^{n-1} x + n(n-1) I_{n-2}$$

If we similarly expand out I_{n-2} as it appears in the above equation, we will have the following:

$$I_n = x \ln^n x - nx \ln^{n-1} x + n(n-1)(x \ln^{n-2} x - (n-2) I_{n-3})$$

Multiplying out the brackets at the end of the right-hand side, we will arrive at the following:

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$$I_n = x \ln^n x - nx \ln^{n-1} x + n(n-1)x \ln^{n-2} x - n(n-1)(n-2)I_{n-3}$$

Comparing this expanded representation of I_n with the entries in our table, we can see that the coefficients of the terms of the form $kx \ln^i x$ start with 1, then follow the pattern $n, n(n-1), n(n-1)(n-2)$ and so on.

Furthermore, we can extend the long expression for I_n as follows:

$$I_n = x \ln^n x - nx \ln^{n-1} x + n(n-1)x \ln^{n-2} x - n(n-1)(n-2)x \ln^{n-3} x + \dots$$

The final term (based on what we observed in our earlier table of specific integrals) will be of the form kx , where k is the same as the coefficient of the term in $x \ln x$ – so that we would expect the last two terms in the above series to be:

$$n(n-1)(n-2) \dots 3.2.x \ln x \pm n(n-1)(n-2) \dots 3.2.1.x$$

where the sign separating the two terms will be a plus sign for even values of n and a minus sign for odd values. (For supporting evidence, take another look at the table.)

Based on this attempt to construct the explicit formula for I_n , I will now suggest a version of this formula using the more concise sigma-notation. My suggestion (which I will proceed to prove in a moment) is as follows:

$$I_n = \sum_{i=0}^n (-1)^{n-i} \frac{n!}{i!} x \ln^i x + C$$

(Note that I have taken care to add the constant of integration C .)

Let' me immediately explain some of the shorthand elements of this expression of the formula. In order to offer some initial justification for this formula, I would remark first that when we think about the expansion of a sigma-notation expression such as:

$$\sum_{i=0}^n a_i$$

we often think of it as ordered in this way:

$$a_0 + a_1 + a_2 + \dots + a_{n-1} + a_n$$

In order to match the pattern established in the table we considered above for the solutions to the integrals I_n for the first six values of n , however, I will in this instance look at the expansion of the sigma-notation expression I have proposed in the reverse order, as follows:

$$a_n + a_{n-1} + a_{n-2} + \dots + a_1 + a_0$$

First, the coefficient $(-1)^{n-i}$ is there to ensure that the signs separating terms in the expanded series alternate in the way we observed in our table. For the highest power of $\ln x$, which is $\ln^n x$, that coefficient will be $(-1)^{n-n} = (-1)^0 = 1$: and we see from our table that

each evaluation of I_n does indeed begin with a positive term. For the next lowest power of $\ln x$, where the value of i will be $n - 1$, the coefficient generated will be $(-1)^{n-(n-1)} = (-1)^1 = -1$. Because of the behaviour of -1 when it is raised to an odd or even power, the signs will continue to alternate happily each time we step down the value of i from n to 0 .

Second, the numeric coefficient of each term is generated by the expression $n!/i!$. The coefficient of the first term, therefore (the term a_n in my discussion of the expansion of the sigma-notation formula), will be $n!/n!$, which is, of course, 1; and this matches the results we saw in the table earlier – the first term, which is always $x \ln^n x$, always has a coefficient of 1.

The formula $n!/i!$ always predicts the coefficients of the subsequent terms in each of the solutions set out in the table. As an example, let's look in detail at the solution:

$$I_4 = x \ln^4 x - 4x \ln^3 x + 12x \ln^2 x - 24x \ln x + 24x$$

We can see that our proposed formula does indeed generate each coefficient in the above expression, as follows:

$$i = 4: \frac{4!}{4!} = 1$$

$$i = 3: \frac{4!}{3!} = 4$$

$$i = 2: \frac{4!}{2!} = 4 \cdot 3 = 12$$

$$i = 1: \frac{4!}{1!} = 4 \cdot 3 \cdot 2 = 24$$

$$i = 0: \frac{4!}{0!} = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

In the last case, we should recall that $0!$ is defined to be equal to 1. Indeed, this fact is what allows our formula to predict reliably that the last two coefficients every solution shown in the table should be the same. Expressed more generally, we note that:

$$\frac{n!}{1!} = \frac{n!}{1} = n!$$

and also that:

$$\frac{n!}{0!} = \frac{n!}{1} = n!$$

so that:

$$\frac{n!}{1!} = \frac{n!}{0!}$$

for all natural numbers n .

Finally, the last part of my proposed sigma-notation formula, $x \ln^i x$, expresses the general form of each term in the solutions we saw in the table, once we take out of consideration the coefficient and the sign – both of which have been dealt with in the preceding discussion.

Having demonstrated that the formula I have put forward accords with the data from our table, it simply remains to provide a formal proof.

The Proof

We will prove the correctness of the sigma-notation formula using mathematical induction, and we will assume the correctness of the reduction formula, obtained in [Part 2 Section 13](#) using integration by parts:

$$I_n = x \ln^n x - n I_{n-1}$$

As the first step, we will show that the formula holds true for the case where $n = 1$. We need to be concerned here with the integral I_1 , and in this case our formula is as follows (omitting the constant of integration to make the rest of the expression clearer):

$$\sum_{i=0}^1 (-1)^{1-i} \frac{1!}{i!} x \ln^i x = (-1)^{1-1} \frac{1!}{1!} x \ln^1 x + (-1)^{0-1} \frac{1!}{0!} x \ln^0 x$$

The right-hand side (which is shown with the higher value of i first, as explained earlier) reduces to the following:

$$(-1)^0 \cdot 1 \cdot x \ln x + (-1)^{-1} \cdot 1 \cdot x = x \ln x - x$$

The final right-hand side here accords exactly with our already-established solution for the integral I_1 . We have therefore proved the first step in our process of induction, namely the case where $n = 1$.

The second stage is to assume that the formula holds good when $n = k$ and try to prove its truth when $n = k + 1$. We are therefore making the assumption that:

$$I_k = \sum_{i=0}^k (-1)^{k-i} \frac{k!}{i!} x \ln^i x$$

I pointed out at the beginning of this proof that we could also safely assume the truth of the reduction formula for I_n . This assumption allows us to say that:

$$I_{k+1} = x \ln^{k+1} x - (k + 1) I_k$$

Substituting the above sigma-notation formula for I_k into this reduction formula, we obtain:

$$I_{k+1} = x \ln^{k+1} x - (k+1) \sum_{i=0}^k (-1)^{k-i} \frac{k!}{i!} x \ln^i x$$

The right-hand side can be rewritten as follows:

$$(-1)^0 \frac{(k+1)!}{(k+1)!} x \ln^{k+1} x + \sum_{i=0}^k (-1)^1 (-1)^{k-i} \frac{(k+1)k!}{i!} x \ln^i x$$

In the above restatement, I have multiplied the first term on the right-hand side by the rather contrived coefficient:

$$(-1)^0 \frac{(k+1)!}{(k+1)!}$$

This is, of course, equal to 1, so it does not change the value of the term to which I have applied it. It has been chosen, however, in order to make our subsequent reasoning more convenient. The minus sign which preceded the sigma-notation expression now appears inside this expression as $(-1)^1 = -1$, which will ensure that the minus sign is applied to each term in the expanded series. I have also taken the coefficient $k+1$ inside the sigma-notation expression, which we are entitled to do by appealing to the distributive law.

Rewriting a little more to produce an equivalent expression, we obtain:

$$(-1)^{(k+1)-(k+1)} \frac{(k+1)!}{(k+1)!} x \ln^{k+1} x + \sum_{i=0}^k (-1)^{(k+1)-i} \frac{(k+1)!}{i!} x \ln^i x$$

Here, we should simply observe that the power of the first instance of -1 has been restated as $(k+1) - (k+1) = 0$. The laws of indices let us say that $(-1)^1 (-1)^{k-i} = (-1)^{(k+1)-i}$, and finally we have shortened the numerator of the rational expression inside the sigma-notation by noting that $(k+1)k! = (k+1)!$.

Since we have taken care to present the first term in the above expression in the appropriate form, we can now combine the two terms into a single sigma-notation expression for I_{k+1} as follows:

$$I_{k+1} = \sum_{i=0}^{k+1} (-1)^{(k+1)-i} \frac{(k+1)!}{i!} x \ln^i x$$

And so we see that having assumed that the formula is true when $n = k$, we have been able to use the reduction formula to prove that it is then also true when $n = k+1$. This completes the second stage of the proof by induction, so that we have now demonstrated that the proposed sigma-notation formula for I_n holds good for all natural-number values of n .

15 A Formula for the Function $B(m, n)$

In [Part 2 Section 13 Example 4](#), we looked at the general integral:

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

which is one form of Euler's beta function. We derived a simple but interesting reduction formula for this definite integral, which was as follows:

$$B(m, n) = \frac{n-1}{m} B(m+1, n-1)$$

I will now briefly investigate the implications of this reduction formula and derive an explicit expression for $B(m, n)$ which does not depend on any other integrals.

To start the ball rolling, let's see what happens when we apply the reduction formula to the function $B(m+1, n-1)$ which appears on the right-hand side of the original formula. We will obtain:

$$B(m+1, n-1) = \frac{n-2}{m+1} B(m+2, n-2)$$

Substituting this back into the original reduction formula gives us:

$$B(m, n) = \frac{n-1}{m} \cdot \frac{n-2}{m+1} B(m+2, n-2)$$

Repeating this process once more will give us:

$$B(m, n) = \frac{n-1}{m} \cdot \frac{n-2}{m+1} \cdot \frac{n-3}{m+2} B(m+3, n-3)$$

Since we observe that the value of the second argument of the function B decrements by 1 at each application of the reduction formula, we can see that we could continue this repeated process until we are left with an expression of the form $B(p, 1)$ on the right-hand side. This would be expected to happen after $n-1$ applications of the reduction formula, so that we would expect to find that $p = m + n - 1$ - since the first argument of the function B will have been *incremented* the same number of times.

When we get to this point, our expression for $B(m, n)$ will look like this:

$$B(m, n) = \frac{n-1}{m} \cdot \frac{n-2}{m+1} \cdot \frac{n-3}{m+2} \cdots \frac{1}{m+n-2} B(m+n-1, 1)$$

Now, we obtained an expression for $B(m, 1)$ in the main text when we were looking at the example on which this discussion is based; it was as follows:

$$B(m, 1) = \frac{1}{m}$$

We can apply this result to $B(m+n-1, 1)$ in order to obtain:

$$B(m+n-1,1) = \frac{1}{m+n-1}$$

Putting this into our earlier expression for $B(m,n)$ turns it into the following:

$$B(m,n) = \frac{n-1}{m} \cdot \frac{n-2}{m+1} \cdot \frac{n-3}{m+2} \cdots \frac{1}{m+n-2} \cdot \frac{1}{m+n-1}$$

All we now have to do is to find neater expressions for the numerator and denominator in this expression after all the fractional factors are multiplied together, and we will be home and dry.

First of all, we consider the numerator. This is fairly simple – it can be abbreviated as $(n-1)!$. The denominator seems at first a harder nut to crack. However, if we rewrite it in reverse order, as follows:

$$(m+n-1)(m+n-2) \dots (m+2)(m+1)m$$

It should be possible to see that if it were to be multiplied by $(m-1)!$, then it would amount to an expanded form of $(m+n-1)!$. This is because:

$$(m-1)! = (m-1)(m-2) \dots 3 \cdot 2 \cdot 1$$

and therefore multiplying the above product expression beginning with $(m+n-1)$ by the additional factor $(m-1)!$ produces the following:

$$(m+n-1)(m+n-2) \dots (m+2)(m+1)m(m-1)(m-2) \dots 3 \cdot 2 \cdot 1$$

We can certainly now abbreviate this to:

$$(m+n-1)(m+n-2) \dots 3 \cdot 2 \cdot 1$$

which is nothing more than the expanded form of $(m+n-1)!$.

This should inspire us to multiply our right-hand side expression “top and bottom” by $(m-1)!$, - which we can do without altering its value. This would bring us to the following equivalent expression:

$$\frac{(n-1)!}{(m+n-1)(m+n-2) \dots (m+2)(m+1)m} \cdot \frac{(m-1)!}{(m-1)!} = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

This gives us a neat and concise expression for $B(m,n)$ and we can now conclude by writing:

$$B(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

which is the desired formula for our function, in a representation without any other integrals.

16 Two Standard Integrals Derived Using Partial Fractions

In [Part 3 Section 1](#), I mentioned two rules of integration which arise from the Method of Partial Fractions. I will demonstrate here the derivation of these two rules, which constitute two additional standard integrals, using symbolic reasoning.

The First Rule

The first of the two rules gives us a formula for the integral:

$$I = \int \frac{1}{(x+a)(x+b)} dx$$

Apart from the fact that the symbols a and b here stand in place of the numbers we encounter in the examples of [Part 3 Section 1](#), the integrand is in the exact form required for the Method of Partial Fractions to be applied, followed by some simple term-by-term integration of the resulting expression in x .

So we will proceed using the methods of [Part 3 Section 1](#), and attempt to restate the integrand in the following form:

$$\frac{1}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

Rewriting the right-hand side using the common denominator $(x+a)(x+b)$, we obtain:

$$\frac{A(x+b)}{(x+a)(x+b)} + \frac{B(x+a)}{(x+a)(x+b)} = \frac{A(x+b) + B(x+a)}{(x+a)(x+b)}$$

We now proceed to equate the numerator on the right-hand side with the (very simple) numerator of our original integrand, as follows:

$$A(x+b) + B(x+a) = 1$$

and we next choose values of x which will allow us to extract from this identity simple equations in A and B .

First, we will set $x = -a$, wwhich will make the factor $x+a$ take a value of zero. We then obtain:

$$A(b-a) = 1 \text{ so that } A = \frac{1}{b-a}$$

Next, we choose the value $x = -b$, so that the factor $x+b$ will become zero. This gives us:

$$B(a-b) = 1 \text{ so that } B = \frac{1}{a-b}$$

Supplementary Examples and Investigations

These values for A and B can now be substituted back into the expression for the decomposition of our integrand, to give us the following identity:

$$\frac{1}{(x+a)(x+b)} = \frac{1}{(b-a)(x+a)} + \frac{1}{(a-b)(x+b)}$$

and we can now attempt to integrate our original expression as follows:

$$I = \int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a} \int \frac{1}{x+a} dx + \frac{1}{a-b} \int \frac{1}{x+b} dx$$

The right-hand side integrates easily using one of the standard rules of integration we derived in [Part 1](#) using the method of substitution to give the following solution:

$$\frac{1}{b-a} \ln|x+a| + \frac{1}{a-b} \ln|x+b| + C$$

This expression can be simplified further if we recall that, as a matter of simple algebra, we can always write:

$$b-a = -(a-b)$$

so that as a consequence we can say that:

$$\frac{1}{b-a} = -\frac{1}{a-b}$$

This equivalence lets us rewrite our solution as follows:

$$\frac{1}{a-b} \ln|x+b| - \frac{1}{a-b} \ln|x+a| + C = \frac{1}{a-b} (\ln|x+b| - \ln|x+a|) + C$$

In order to simplify this result as far as we can using symbols, we finally quote the rule of logarithms which states that:

$$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$$

This lets us simplify the material in brackets in the last version of our solution, so that we can now state a final version, which also constitutes a statement of the rule of integration we have been demonstrating:

$$I = \int \frac{1}{(x+a)(x+b)} dx = \frac{1}{a-b} \ln \left| \frac{x+b}{x+a} \right| + C$$

An Alternative Statement of the First Rule

There is actually an alternative statement of this rule, which you may encounter. I will derive the alternative statement here, and then prove that it is actually identical at the end of the day to the version set out above.

Supplementary Examples and Investigations

In order to see this alternative version, we must follow through the procedure we have adopted above, but when we arrive at the following` version of the solution to our integral:

$$\frac{1}{b-a} \ln|x+a| + \frac{1}{a-b} \ln|x+b| + C$$

instead of using the identity:

$$\frac{1}{b-a} = -\frac{1}{a-b}$$

to simplify the result further, we will instead use the equally valid identity:

$$\frac{1}{a-b} = -\frac{1}{b-a}$$

This lets us restate the solution as follows:

$$\frac{1}{b-a} \ln|x+a| - \frac{1}{b-a} \ln|x+b| + C = \frac{1}{b-a} (\ln|x+a| - \ln|x+b|) + C$$

We see here that we have chosen to retain the expression $b-a$ in our simplification rather than choosing $a-b$, as we did in the main exposition above.

Finally, appealing once more to the rule of logarithms which allowed us to reach the final stage earlier, we now arrive at the alternative statement of our rule of integration, which is as follows:

$$I = \int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a} \ln \left| \frac{x+a}{x+b} \right| + C$$

This looks initially to be quite different from the first version of our rule. However, closer investigation will prove more reassuring.

We first note a consequence of the rule of logarithms which we have been calling to our aid at the last stage of simplifying the result of our integration. To begin with, we can certainly state that:

$$\ln a - \ln b = \ln \left(\frac{a}{b} \right) \text{ and also that } \ln b - \ln a = \ln \left(\frac{b}{a} \right)$$

However, we also relied on the rule of algebra which states that:

$$b-a = -(a-b)$$

Now since $\ln a$ and $\ln b$ are both perfectly good real numbers, to which this rule of algebra will also apply, we can deduce from it that:

$$\ln b - \ln a = -(\ln a - \ln b)$$

Recalling that each of the differences of two logarithms in this identity can also be expressed as the logarithm of a quotient, we come to the interesting conclusion that it must be true to say:

$$\ln\left(\frac{b}{a}\right) = -\ln\left(\frac{a}{b}\right)$$

Readers with a strong familiarity with logarithms and their behavior will no doubt dismiss this with an air of superiority as a trivial and well-known property of logarithms, but I have taken some trouble to justify it for the benefit of those readers without such wide experience.

We must immediately note that a consequence of this identity which affects our current concerns more closely is the following:

$$\ln\left|\frac{x+b}{x+a}\right| = -\ln\left|\frac{x+a}{x+b}\right|$$

Remembering also that:

$$\frac{1}{a-b} = -\frac{1}{b-a}$$

we can finally bring together the facts we have noted here to be able to state with confidence that:

$$\frac{1}{a-b} \ln\left|\frac{x+b}{x+a}\right| = \left(-\frac{1}{b-a}\right) \left(-\ln\left|\frac{x+a}{x+b}\right|\right) = \frac{1}{b-a} \ln\left|\frac{x+a}{x+b}\right|$$

The final step here is made by the cancelling out of the minus signs in the product of two factors in the intermediate step – this amounts to relying on an identity of the form:

$$(-a)(-b) = ab$$

We therefore see – leaving out of consideration the constant of integration – that the two versions of the solution to our integral turn out to be equal, and the two versions of our rule of integration therefore turn out to be equivalent.

The Second Rule

The second rule of integration I want to exhibit here can be regarded as a special case of the first rule, once we have performed a little algebra on the integrand involved. The general integral we will consider here is as follows:

$$I = \int \frac{1}{x^2 - a^2} dx$$

In order to turn this integral into something more recognizable in the current context, we must apply the algebraic identity which lets us state that:

$$x^2 - a^2 = (x + a)(x - a)$$

This means that our integral can be restated as:

$$I = \int \frac{1}{(x + a)(x - a)} dx$$

This integral is now indeed a special case of the one we dealt with earlier in this section, with the simple substitution $b = -a$.

We can therefore write down its solution fairly easily by recalling the statement of our first Partial Fractions rule of integration:

$$\int \frac{1}{(x + a)(x + b)} dx = \frac{1}{a - b} \ln \left| \frac{x + b}{x + a} \right| + C$$

and making the substitution $b = -a$ in this rule.

I will take a moment to point out the effect of this substitution on the constant coefficient on the right-hand side. The substitution changes this coefficient into:

$$\frac{1}{a - (-a)} = \frac{1}{a + a} = \frac{1}{2a}$$

The change to the logarithmic function is quite easy to see, and it is altered to the following expression:

$$\ln \left| \frac{x - a}{x + a} \right|$$

Putting these together, we can now state the second rule of integration which is a consequence of the use of Partial Fractions:

$$I = \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + C$$

17 An Investigation of the More General Partial Fractions Problem

Having derived two standard integrals using the Method of Partial Fractions in the previous section, I will devote some space here to an investigation of whether a similar job can be done for what one might regard as the “general” integral which is approachable using partial fractions. This “general” integral can be expressed as follows using symbols:

$$I = \int \frac{ax + b}{(cx + d)(ex + f)} dx$$

We have in this integral no less than six symbolic constants, so we should stand ready for some serious symbol manipulation. As always in these circumstances, the best advice is to keep our heads and stick rigorously to the rules of algebra.

Supplementary Examples and Investigations

So let's see how far we can get. As in the previous section, it will be our first task to restate the integrand as follows:

$$\frac{ax + b}{(cx + d)(ex + f)} = \frac{A}{cx + d} + \frac{B}{ex + f}$$

Converting the right-hand side using the common denominator $(cx + d)(ex + f)$ we will get:

$$\frac{A(ex + f)}{(cx + d)(ex + f)} + \frac{B(cx + d)}{(cx + d)(ex + f)} = \frac{A(ex + f) + B(cx + d)}{(cx + d)(ex + f)}$$

Equating the numerator on the right-hand side with that of our original integrand gives us the following equation:

$$A(ex + f) + B(cx + d) = ax + b$$

At this point, things become a little more complicated than previously. First of all, we want to choose a value for x which will make the factor $cx + d$ take the value zero, thus allowing us to end up with an equation for A alone. We therefore require:

$$cx + d = 0 \text{ so that } cx = -d \text{ and thus } x = -\frac{d}{c}$$

This converts our equation of numerators into the following:

$$A\left(f - \frac{de}{c}\right) = b - \frac{ad}{c} \text{ so that } A = \frac{\left(b - \frac{ad}{c}\right)}{\left(f - \frac{de}{c}\right)}$$

This is more than a little ugly and cumbersome. We can do a little better, however, by transforming the numerator and denominator on the right-hand side above into single fractions with a common denominator of c , as follows:

$$b - \frac{ad}{c} = \frac{bc - ad}{c} \text{ and } f - \frac{de}{c} = \frac{cf - de}{c}$$

In the fractional expression we obtained for A , the "factor" $1/c$ can be cancelled from the numerator and denominator, leaving us with the following somewhat neater expression for A :

$$A = \frac{bc - ad}{cf - de}$$

We can derive a formula for B in a very similar way. First, recall the equation we obtained by equating numerators:

$$A(ex + f) + B(cx + d) = ax + b$$

Supplementary Examples and Investigations

This time, we need to choose a value for x which will make the factor which multiplies A take a zero value – that is to say, we need to set:

$$ex + f = 0 \text{ so that } ex = -f \text{ and thus } x = -\frac{f}{e}$$

The equation we recalled a few lines ago then becomes:

$$B \left(d - \frac{cf}{e} \right) = b - \frac{af}{e} \text{ so that } B = \frac{\left(b - \frac{af}{e} \right)}{\left(d - \frac{cf}{e} \right)}$$

As before, we can re-express the numerator and denominator on the right-hand side as two single fractions with a denominator of e as follows:

$$b - \frac{af}{e} = \frac{be - af}{e} \text{ and } d - \frac{cf}{e} = \frac{de - cf}{e}$$

We can then cancel the “factor” of $1/e$ from both numerator and denominator to reach the following neater expression for B :

$$B = \frac{be - af}{de - cf}$$

Having found expressions for both A and B , we can now restate our integrand as the sum of two fractional expressions. In order to avoid a cluttered and hard-to-grasp expression, I will here show the result of our efforts in the following form:

$$A \cdot \frac{1}{cx + d} + B \cdot \frac{1}{ex + f}$$

This form will also make it easier to see what has to happen when we proceed to the actual integration. Our symbolic integrand can be rewritten as follows:

$$\frac{ax + b}{(cx + d)(ex + f)} = \frac{bc - ad}{cf - de} \cdot \frac{1}{cx + d} + \frac{be - af}{de - cf} \cdot \frac{1}{ex + f}$$

(We should perhaps pause for a moment to contemplate what would seem to be the general solution to partial fractions decompositions of expressions which take the form shown on the left-hand side above. We might also notice that this kind of decomposition will not work in cases where $cf = de$, since this would make the denominator of each of our constant factors zero.)

We are almost finished. We can now proceed to perform the integration of the expression we have been working on, noting that it will be possible to split it into two easier integrals as follows:

$$\int \frac{ax + b}{(cx + d)(ex + f)} dx = \frac{bc - ad}{cf - de} \int \frac{1}{cx + d} dx + \frac{be - af}{de - cf} \int \frac{1}{ex + f}$$

Each of the right-hand side integrals here can be evaluated using the rule of integration we derived using the method of substitution in [Part 1 Section 7](#). The relevant rule is this one:

$$\int \frac{1}{ax + b} dx = \frac{1}{a} \ln|ax + b| + C$$

Applying this rule (noting carefully that we must take account of the coefficient of $1/a$ which appears on the right-hand side) we can now write down the solution to our general integral:

$$I = \int \frac{ax + b}{(cx + d)(ex + f)} dx = \frac{bc - ad}{c(cf - de)} \ln|cx + d| + \frac{be - af}{e(de - cf)} \ln|ex + f| + C$$

Verification of the Symbolic Solution

Since this turns out to be quite a complicated formula for the solution, it might be reassuring to check that it predicts correctly at least one of the examples we evaluated by hand in [Part 3 Section 1](#). Let's take [Example 2](#) as our test. The integral evaluated there, with its solution, was as follows:

$$I = \int \frac{7x + 5}{(x - 1)(x + 2)} dx = 4 \ln|x - 1| + 3 \ln|x + 2| + C$$

The integrand is an instance of the general integrand we have been investigating here, with the following values for the symbolic constants:

$$a = 7, b = 5, c = 1, d = -1, e = 1, f = 2$$

We should be able to see immediately (“by inspection” as the mathematical phrase would have it) that the arguments of the logarithmic functions enclosed in modulus brackets already conform to the pattern predicted by our symbolic solution. It therefore just remains for us to prove the following two identities, and we will have verified that our general formula correctly predicts the solution to the example we have taken from the main text (these identities are obtained by equating the coefficients of the logarithmic functions):

$$\frac{bc - ad}{c(cf - de)} = 4 \text{ and } \frac{be - af}{e(de - cf)} = 3$$

Substituting the values for our symbolic constants into the left-hand side of the first of these, we have:

$$\frac{5 \cdot 1 - 7 \cdot (-1)}{1 \cdot (1 \cdot 2 - (-1) \cdot 1)} = \frac{5 + 7}{1 \cdot (2 + 1)} = \frac{12}{3} = 4$$

Moving on to the second required identity, its left-hand side yields:

$$\frac{5.1 - 7.2}{1.((-1).1 - 1.2)} = \frac{5 - 14}{1.((-1) - 2)} = \frac{-9}{-3} = 3$$

and so we see that our symbolic coefficients do in fact also predict the coefficients of the logarithmic functions in the solution to our concrete example. This provides some evidence for the correctness of the symbolic general solution we have derived here.

A Condition for a Solution which Combines the Two Logarithmic Functions

In [Part 3 Section 1 Example 1](#), we evaluated an integral using partial fractions and were eventually able to reduce the solution to an expression of the form:

$$k \ln \left(\frac{p}{q} \right)$$

The second rule of integration established in the preceding section also produced a solution which could be reduced to this form.

It would be interesting to extend the current investigation a little to see if we can establish a condition on the symbolic constants in our general partial fractions integral which would allow us to predict instances which would yield a solution in this more compact form.

We need to note first of all that in the cases where we have obtained solutions of the desired form, we have always been able to finish our work by simplifying our solution as follows:

$$k \ln p - k \ln q = k(\ln p - \ln q) = k \ln \left(\frac{p}{q} \right)$$

We can see that this chain of equalities required the coefficients of the two logarithmic functions which the integration step gave us to be equal and separated by a minus sign.

In terms of the coefficients of the general solution we have just worked out, this would mean that the following would have to be true:

$$\frac{bc - ad}{c(cf - de)} = -\frac{be - af}{e(de - cf)}$$

We can immediately remark that the denominators of these two expressions each contain a bracketed factor; in the first expression, this has the form $p - q$, whereas in the second it has the form $q - p$ - in other words, the bracketed factors contain the same terms, but in reverse order. If we make the further observation, using simple algebra, that $p - q = -(q - p)$, we should be able to see that we can remove the minus sign on the right-hand side by simply reversing the order of terms in the bracketed factor in the denominator. This means that our equality above is equivalent to the following restatement:

$$\frac{bc - ad}{c(cf - de)} = \frac{be - af}{e(cf - de)}$$

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Here, both sides share an identical factor in their denominators. We can therefore multiply both sides by this factor, which is $cf - de$, to obtain the following equivalent identity:

$$\frac{bc - ad}{c} = \frac{be - af}{e}$$

To go a step further, we can first multiply both sides by c , and then multiply both sides by e , to get the following equivalent identity:

$$e(bc - ad) = c(be - af)$$

Multiplying out the brackets produces the following:

$$bce - ade = bce - acf$$

Subtracting bce from both sides, we have:

$$-ade = -acf$$

And a final step of dividing both sides by $-a$ produces the surprisingly brief equivalent identity:

$$de = cf$$

However (and this is a really big “however”) if this equivalence is true, then we will have the consequences:

$$cf - de = 0 \text{ and } de - cf = 0$$

which would make the constant coefficients predicted by our general formula impossible to calculate since in each case the denominator would be zero!

We have arrived in our reasoning at a contradiction. This means that we are forced to the (actually rather interesting) conclusion that in the general partial fractions integral, if a is non-zero, there are actually *no* cases where we can integrate to a solution comprising a single logarithmic function!

The Cases Where $a = 0$

It just remains to investigate what we can say about the cases where the coefficient a in our standard expression is equal to zero: these are the cases where there is no x term in the numerator of our expression.

We can immediately make one useful statement: that is that any integrand of the form:

$$\frac{k}{(x + a)(x + b)}$$

will integrate to give just one single logarithmic function. We can be confident about this because in the previous section we evaluated the integral:

$$\int \frac{1}{(x+a)(x+b)} dx$$

and found that the solution had the desired form. The assertion I made a few lines ago can be seen to be true since we can always be sure that:

$$\int \frac{k}{(x+a)(x+b)} dx = k \int \frac{1}{(x+a)(x+b)} dx$$

We therefore just need to look at the following general integral:

$$\int \frac{k}{(ax+b)(cx+d)} dx$$

As noted a moment ago, we can always move the constant numerator k outside the integral sign. So we only really need to consider the following integral:

$$I = \int \frac{1}{(ax+b)(cx+d)} dx$$

The reasoning involved in producing a solution for this general integral will be similar to that which we followed at the outset of this section. However, in order to avoid any pitfalls arising from the absence of an x term in the numerator of the integrand, I will go through it again in brief.

We wish, of course, first to restate our integrand as follows:

$$\frac{1}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$$

As before, we convert the right-hand side using the common denominator $(ax+b)(cx+d)$ as follows:

$$\frac{A(cx+d)}{(ax+b)(cx+d)} + \frac{B(ax+b)}{(ax+b)(cx+d)} = \frac{A(cx+d) + B(ax+b)}{(ax+b)(cx+d)}$$

Equating the right-hand side numerator with the numerator of our original integrand gives us:

$$A(cx+d) + B(ax+b) = 1$$

To find a suitable value for a , we must find a value for x which will make $ax+b$ equal to zero. This requires:

$$ax+b=0 \text{ so that } ax = -b \text{ and } x = -\frac{b}{a}$$

With this value for x , our equation of numerators becomes:

$$A\left(d - \frac{bc}{a}\right) = 1 \text{ or alternatively } A\left(\frac{ad - bc}{a}\right) = 1 \text{ so that } A = \frac{a}{ad - bc}$$

Adopting an analogous approach to find a value for B , we must make the factor which multiplies A take a zero value, so that:

$$cx + d = 0 \text{ so that } cx = -d \text{ and thus } x = -\frac{d}{c}$$

Once again, using this value for x in our equation of numerators, we have:

$$B\left(b - \frac{ad}{c}\right) = 1 \text{ or alternatively } B\left(\frac{bc - ad}{c}\right) = 1 \text{ so that } B = \frac{c}{bc - ad}$$

Putting these values for A and B in place, we are now able to restate our integrand as follows:

$$\frac{1}{(ax + b)(cx + d)} = \frac{a}{ad - bc} \cdot \frac{1}{ax + b} + \frac{c}{bc - ad} \cdot \frac{1}{cx + d}$$

This means that we can now perform our job of integration by means of the following equivalence:

$$\int \frac{1}{(ax + b)(cx + d)} dx = \frac{a}{ad - bc} \int \frac{1}{ax + b} dx + \frac{c}{bc - ad} \int \frac{1}{cx + d} dx$$

Using the same rule of integration as before, this integrates to give the following solution:

$$I = \int \frac{1}{(ax + b)(cx + d)} dx = \frac{1}{ad - bc} \ln|ax + b| + \frac{1}{bc - ad} \ln|cx + d| + C$$

Note that in this solution, the numerators of both the coefficients of the logarithmic functions are found to be 1. This is because the values a and c in the numerators of the constants which we took outside the integral signs can be cancelled due to the fact that the rule of integration we are using here lets us state that:

$$\int \frac{1}{ax + b} dx = \frac{1}{a} \ln|ax + b| + C \text{ and } \int \frac{1}{cx + d} dx = \frac{1}{c} \ln|cx + d| + C$$

Using the same reasoning as before, we can be sure that the general solution we have sketched out above will reduce to one single logarithmic function whenever the coefficients of the two functions in our general solution satisfy the following equivalence:

$$\frac{1}{ad - bc} = -\frac{1}{bc - ad}$$

However, using the general algebraic rule which states that:

$$a - b = -(b - a)$$

we can immediately make the following statement about the second of our two coefficients:

$$\frac{1}{bc - ad} = -\frac{1}{ad - bc}$$

We therefore see that our equivalence actually reduces to the following:

$$\frac{1}{ad - bc} = \frac{1}{ad - bc}$$

which is, of course, *always* true!

We can therefore conclude that it is *always* the case that integrals of the form:

$$\int \frac{1}{(ax + b)(cx + d)} dx$$

evaluate to a solution comprising one single logarithmic function. It will be important to add, however, that our reasoning can only be relied upon as long as the expression $ad - bc$ is non-zero; otherwise the coefficients in our general solution make no sense, and cannot be assigned any values.

Conclusion

We can conclude this section by stating what would seem to be the conclusion of our investigation. We set out to examine the conditions which might have to be satisfied for an integral of the following form:

$$I = \int \frac{ax + b}{(cx + d)(ex + f)} dx$$

to evaluate to a solution of this form:

$$k \ln\left(\frac{p}{q}\right)$$

We concluded that it all depends on the value taken by the placeholder a : if $a \neq 0$, then the integral will not evaluate to the desired form; however, if $a = 0$, then the integral will *always* evaluate to the desired form.

18 Integration of $29/(x^2 - 5)$ Using Partial Fractions

In the final example of [Part 3 Section 5](#), we had to integrate a remainder term which was left over after carrying out polynomial division. The integral in question was as follows:

$$I_1 = \int \frac{29}{x^2 - 5} dx$$

In the text, the integration was carried out by quoting a standard integral which in turn was derived by means of the Method of Partial Fractions. I present here the explicit procedure,

without appeal to a standard integral for those who would like to see it, and as a further worked example of the kind contained in [Part 3 Section 1](#).

We first note that the denominator of the integrand here need not be treated as an irreducible quadratic. We can use the familiar rule of algebra which tells us that:

$$a^2 - b^2 = (a - b)(a + b)$$

To rewrite our integrand as follows:

$$\frac{29}{(x - \sqrt{5})(x + \sqrt{5})}$$

In this form, the job of integrating our expression can be carried out using an approach based on partial fractions. We will try to rewrite the expression in the following form:

$$\frac{A}{x - \sqrt{5}} + \frac{B}{x + \sqrt{5}}$$

Writing our desired form using the common denominator $(x - \sqrt{5})(x + \sqrt{5})$ will give us the following version:

$$\frac{A(x + \sqrt{5}) + B(x - \sqrt{5})}{(x - \sqrt{5})(x + \sqrt{5})}$$

Equating the numerator with that of our original rational expression gives us the following equation:

$$A(x + \sqrt{5}) + B(x - \sqrt{5}) = 29$$

We should be able to find values for the placeholders A and B by choosing suitable values of x . Let's start by producing an equation in A alone by choosing the value $x = \sqrt{5}$. This gives us the following equation:

$$2\sqrt{5}A = 29 \text{ so that } A = \frac{29}{2\sqrt{5}}$$

If we now choose the value $x = -\sqrt{5}$, we will obtain the following equation in B alone:

$$-2\sqrt{5}B = 29 \text{ so that } B = -\frac{29}{2\sqrt{5}}$$

We have therefore been able to rewrite our integrand in the following form:

$$\frac{29}{2\sqrt{5}} \cdot \frac{1}{x - \sqrt{5}} - \frac{29}{2\sqrt{5}} \cdot \frac{1}{x + \sqrt{5}}$$

Our integral is then restated in the following way:

$$\int \frac{29}{x^2 - 5} dx = \frac{29}{2\sqrt{5}} \int \frac{1}{x - \sqrt{5}} dx - \frac{29}{2\sqrt{5}} \int \frac{1}{x + \sqrt{5}} dx$$

Using the standard integral with which we have become familiar when integrating with the aid of partial fractions, the solution is as follows:

$$\frac{29}{2\sqrt{5}} \ln|x - \sqrt{5}| - \frac{29}{2\sqrt{5}} \ln|x + \sqrt{5}| + C$$

Noting that the two logarithmic functions share the common coefficient $29/2\sqrt{5}$, and taking advantage of the rules of logarithms, we can rearrange our solution into the following form:

$$I_1 = \frac{29}{2\sqrt{5}} \ln \left| \frac{x - \sqrt{5}}{x + \sqrt{5}} \right| + C$$

which, as we would expect, is the solution our standard integral provided in the main text.

19 A Useful Standard Integral for Some Polynomial Division Problems

In [Part 3 Section 5](#), we integrated some rational functions using a shorthand version of polynomial division to pit them into a form which was relatively easy to deal with. In two cases, the final integration stage required us to evaluate an integral of the form:

$$\int \frac{px + q}{x^2 + bx + c} dx$$

We did this by completing the square for the denominator, using substitution, and finally if necessary splitting the resulting integral in terms of u into two separate parts which could be evaluated using rules already derives elsewhere in this book.

It would obviously be very useful to have a general formula for the solution to an integral of this type, just as we had one which could be used to deal with remainder terms after polynomial division by a linear denominator. I will set out here the reasoning which will lead us to such a formula.

As we did in [Part 3 Section 5](#), we will begin by completing the square for the quadratic denominator of our integrand. The formula for this, which is derived in [Appendix 3 Section 1](#), gives us the following equivalence:

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)$$

We will immediately adopt an abbreviation which will make things easier: the second bracketed term on the right-hand side of the above equation contains nothing but constants, so that it is itself a constant. We will abbreviate it as follows:

$$k_1 = c - \frac{b^2}{4}$$

Our integral has now become:

$$\int \frac{px + q}{\left(x + \frac{b}{2}\right)^2 + k_1} dx$$

Our next step will be to make the substitution:

$$u = x + \frac{b}{2}$$

This gives us the following consequences:

$$\frac{du}{dx} = 1 \text{ so that } dx = du$$

and:

$$x = u - \frac{b}{2} \text{ so that } px + q = pu - \frac{bp}{2} + q$$

Let's introduce a second constant abbreviation, as follows:

$$k_2 = \frac{bp}{2} - q \text{ so that } px + q = pu - k_2$$

We can substitute all these useful facts back into our integral to obtain this integral in terms of u :

$$\int \frac{pu - k_2}{u^2 + k_1} dx$$

We will evaluate this integral by dividing it into two parts and taking any constants we can outside the integral sign, turning it into the following:

$$p \int \frac{u}{u^2 + k_1} du - k_2 \int \frac{1}{u^2 + k_1} du$$

The two integrals in the above expression can be named using the following abbreviations:

$$I_1 = \int \frac{u}{u^2 + k_1} du \text{ and } I_2 = \int \frac{1}{u^2 + k_1} du$$

The complete integral in terms of u will then be obtained using the formula:

$$pI_1 - k_2I_2$$

Both of these integrals are very similar to standard integrals referenced repeatedly in the course of this book. The first of the two, I_1 , is similar the following integral:

$$\int \frac{x}{x^2 + 1} dx = \frac{1}{2} \ln(x^2 + 1) + C$$

Like this standard integral, the integrand in I_1 contains a function-derivative pair multiplied by a constant; in other words, it has the form $kg'(f(u))f'(u)$, where:

$$k = \frac{1}{2}, f(u) = u^2, f'(u) = 2u$$

$$g(u) = \ln|u + k_1|, g'(u) = \frac{1}{u + k_1} \text{ and } g'(f(u)) = \frac{1}{u^2 + k_1}$$

Using the formula describing integration by substitution, we can then write down the solution as the expression $kg(f(u)) + C$, so we can conclude that:

$$I_1 = \frac{1}{2} \ln|u^2 + k_1| + C$$

The second integral, I_2 , is an instance of the following standard integral:

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{1}{a} x + C$$

where $a = \sqrt{k_1}$. Its solution is therefore as follows:

$$I_2 = \frac{1}{\sqrt{k_1}} \tan^{-1} \frac{1}{\sqrt{k_1}} u + C$$

Using the formula we set out earlier to obtain the complete integral in terms of u from the two separate parts we have been considering, we can now write down its solution as follows:

$$\frac{p}{2} \ln|u^2 + k_1| - \frac{k_2}{\sqrt{k_1}} \tan^{-1} \frac{1}{\sqrt{k_1}} u + C$$

It's interesting to note that the abbreviated constant k_1 from the denominator of the integrand in terms of u appears three times in this preliminary version of our solution, while the constant k_2 only shows up once, as a coefficient of the inverse tangent term. Having noted, then, let's move straight on to reverse the substitution. We recall that this was:

$$u = x + \frac{b}{2}$$

so that our solution in terms of x is:

$$\frac{p}{2} \ln \left| \left(x + \frac{b}{2} \right)^2 + k_1 \right| - \frac{k_2}{\sqrt{k_1}} \tan^{-1} \frac{1}{\sqrt{k_1}} \left(x + \frac{b}{2} \right) + C$$

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We now have the job of unpacking the abbreviation constants k_1 and k_2 . We will first remind ourselves briefly of their definitions, which were:

$$k_1 = c - \frac{b^2}{4} \text{ and } k_2 = \frac{bp}{2} - q$$

To begin, we will address the argument of the logarithmic function inside the modulus brackets. Multiplying out the bracketed squared term and substituting the definition of the constant k_1 gives us:

$$\left(x + \frac{b}{2}\right)^2 + k_1 = x^2 + bx + \frac{b^2}{4} + c - \frac{b^2}{4} = x^2 + bx + c$$

which is, interestingly enough, the denominator of our original integrand in x .

We are now faced with expanding out the abbreviations in the second, inverse tangent, term of our solution. Let's start with the coefficient; this is:

$$\frac{k_2}{\sqrt{k_1}}$$

In order to handle this more easily, we will convert our expressions for the two abbreviation constants into two single rational expressions, each with its own common denominator. These expressions are:

$$k_1 = \frac{4c - b^2}{4} \text{ and } k_2 = \frac{bp - 2q}{2}$$

The square root of k_1 can then be written as follows:

$$\sqrt{k_1} = \sqrt{\frac{4c - b^2}{4}} = \frac{\sqrt{4c - b^2}}{2}$$

We therefore now have the unpacked version of the coefficient as follows:

$$\frac{k_2}{\sqrt{k_1}} = \frac{bp - 2q}{2} \cdot \frac{2}{\sqrt{4c - b^2}} = \frac{bp - 2q}{\sqrt{4c - b^2}}$$

Having dealt with this, the coefficient of the bracketed element of the inverse tangent's argument is somewhat easier. We know that:

$$\sqrt{k_1} = \frac{\sqrt{4c - b^2}}{2}$$

So that it becomes a very simple consequence of this to be able to say that:

$$\frac{1}{\sqrt{k_1}} = \frac{2}{\sqrt{4c - b^2}}$$

We can now substitute the results of all these unpacking operations into our preliminary solution in terms of x to obtain the following final result, which we can use to create a standard integral formula which will be very helpful when integrating rational expressions with a quadratic denominator and which require the application of polynomial division. Our solution is:

$$I = \frac{p}{2} \ln|x^2 + bx + c| - \frac{bp - 2q}{\sqrt{4c - b^2}} \tan^{-1} \frac{2}{\sqrt{4c - b^2}} \left(x + \frac{b}{2}\right) + C$$

We ought not to leave this investigation without asking what might happen if the expression $4c - b^2$ turned out to be negative, in which case we would not be able to obtain its square root within the real number system. We would also be unable to use this formula if $4c - b^2$ were equal to zero, since this would entail a division by zero.

The condition which must be satisfied in order to use this formula is therefore:

$$4c - b^2 > 0$$

Multiplying both sides by -1 and reversing the inequality as a consequence of the use of a negative coefficient, we can see that this is equivalent to the following condition:

$$b^2 - 4c < 0$$

This is, of course simply the condition that the discriminant of the monic quadratic in our original denominator should be negative; in other words, it is the condition that that quadratic should have no real roots. We therefore see that our formula can only be applied when the quadratic in the denominator of the integrand has no real roots – in other words, when it is an *irreducible* quadratic.

We ought, before closing this investigation, to check that the formula we have derived for the solution of this particular type of integral does indeed yield the same results as those we obtained by going through the integration process explicitly in the two relevant examples in [Part 3 Section 5](#).

[Example 2](#) of that section included the following integral as part of our working:

$$I_1 = \int \frac{3x + 1}{x^2 + 2x + 2} dx = \frac{3}{2} \ln(x^2 + 2x + 2) - 2 \tan^{-1}(x + 1) + C$$

In the integrand, the following constant coefficients are present, using our chosen model for expressions of this type:

$$p = 3, q = 1, b = 2 \text{ and } c = 2$$

We can now confirm that our formula makes the correct prediction of the result obtained in [Part 3](#), by noting the following equivalences:

$$\frac{p}{2} = \frac{3}{2}$$

$$x^2 + bx + c = x^2 + 2x + 2$$

$$\frac{bp - 2q}{\sqrt{4c - b^2}} = \frac{2.3 - 2.1}{\sqrt{4.2 - 4}} = \frac{6 - 2}{\sqrt{4}} = \frac{4}{2} = 2$$

$$\frac{2}{\sqrt{4c - b^2}} = \frac{2}{\sqrt{4.2 - 4}} = \frac{2}{\sqrt{4}} = \frac{2}{2} = 1$$

$$x + \frac{b}{2} = x + 1$$

All of these equivalences together show that our formula gives the same result as the explicit integration process given in the main text.

Finally, let's look at the interesting case of [Example 5](#). The following integral arose in the course of our working:

$$I_1 = \int \frac{29}{x^2 - 5} dx = \frac{29}{2\sqrt{5}} \ln \left| \frac{x - \sqrt{5}}{x + \sqrt{5}} \right| + C$$

Does our formula work in this case as well? The solution we obtained in the main text doesn't really look quite like the solution in the example we just considered. To find out more, let's first set down the values of the constant coefficients in the integrand; these are:

$$p = 0, q = 29, b = 0 \text{ and } c = -5$$

We immediately note that two of our constants have zero values in this example, so we might expect a significant impact on our workings. However, the most important conclusion from this list of constants is the following:

$$b^2 - 4c = 0 - 4(-5) = 20 > 0$$

The discriminant of the quadratic denominator is greater than zero, so that it has real roots, and is not irreducible. Our formula therefore does not apply in this case.

We might already have noticed that the denominator can easily be factorised as follows:

$$x^2 - 5 = (x - \sqrt{5})(x + \sqrt{5})$$

The integration can therefore be easily carried out using one of the rules derived from the use of partial fractions – and this is exactly what we did in the main text. See also the preceding section in this appendix for an alternative evaluation of the integral in question.

20 The Integral of $1/\sqrt{x^2 + a^2}$ as a Standard Integral

In [Part 3 Section 7 Example 7](#) (and actually before that as well – in [Part 1 Section 11 Example 2](#)) we uncovered an instance of the following standard integral:

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln |x + \sqrt{x^2 + a^2}| + C$$

I will give a derivation here by explicitly evaluating the symbolic integral on the left-hand side, rather than simply appealing to the Fundamental Theorem of Calculus as we did in the main text. This will hopefully help to show why the solution to the integral is so apparently unexpected.

We will therefore evaluate:

$$I = \int \frac{1}{\sqrt{x^2 + a^2}} dx$$

We first remark that the integrand contains the square root expression:

$$\sqrt{x^2 + a^2}$$

which we learned in [Part 1 Section 11](#) indicates that it will respond to the use of tangent-substitution. Our substitution will therefore be:

$$x = a \tan u$$

Differentiating, we obtain:

$$\frac{dx}{du} = a \sec^2 u \text{ so that } dx = a \sec^2 u \, du$$

Our integral in terms of u therefore becomes:

$$\int \frac{1}{\sqrt{a^2 \tan^2 u + a^2}} \cdot a \sec^2 u \, du = \int \frac{a \sec^2 u}{\sqrt{a^2 (\tan^2 u + 1)}} du$$

The right-hand side above can be rewritten as follows by dividing up the square root in the denominator:

$$\int \frac{a \sec^2 u}{\sqrt{a^2} \sqrt{\tan^2 u + 1}} du = \int \frac{a \sec^2 u}{a \sqrt{\tan^2 u + 1}} du = \int \frac{\sec^2 u}{\sqrt{\tan^2 u + 1}} du$$

A trigonometric identity will come to our aid at this point – a version of the identity which connects $\tan x$ and $\sec x$. This identity, in terms of u , which is what we require here, is:

$$\sec^2 u = \tan^2 u + 1$$

and it allows us to rewrite the integral in terms of u as follows:

$$\int \frac{\sec^2 u}{\sqrt{\sec^2 u}} du = \int \frac{\sec^2 u}{\sec u} du = \int \sec u du$$

This has now become a lot simpler: we know the integral of $\sec x$ as a standard integral (I give a justification of the standard integral in [Section 6](#) of this appendix). Our solution in terms of u is therefore:

$$\ln|\sec u + \tan u| + C$$

Reversing the substitution, we first note that we can rearrange the substitution formula to get the following expression for u in terms of x :

$$\frac{x}{a} = \tan u \text{ so that } u = \tan^{-1} \frac{x}{a}$$

and we can use this to arrive at the following solution in terms of x :

$$\ln \left| \sec \tan^{-1} \frac{x}{a} + \tan \tan^{-1} \frac{x}{a} \right| + C$$

Examining the two compound terms inside the modulus brackets, we note first that:

$$\tan \tan^{-1} \frac{x}{a} = \frac{x}{a}$$

and second, that (as derived in detail and listed as a standard trigonometric identity in [Appendix 4 Section 4](#)):

$$\sec \tan^{-1} \frac{x}{a} = \sqrt{\frac{x^2}{a^2} + 1} = \sqrt{\frac{x^2 + a^2}{a^2}} = \frac{1}{a} \sqrt{x^2 + a^2}$$

These two equivalences mean that our solution in x can be written as follows:

$$I = \ln \left| \frac{1}{a} \sqrt{x^2 + a^2} + \frac{x}{a} \right| + C = \ln \left| \frac{\sqrt{x^2 + a^2} + x}{a} \right| + C$$

We can now use the rule of logarithms which states that:

$$\ln \left(\frac{a}{b} \right) = \ln a - \ln b$$

to restate the right-hand side above as follows:

$$\ln \left| \sqrt{x^2 + a^2} + x \right| - \ln|a| + C$$

This allows us to “restate the constant of integration”, as we have done before on several occasions in the main text to write our solution as:

$$\ln \left| \sqrt{x^2 + a^2} + x \right| + C' \text{ where } C' = C - \ln|a|$$

We have taken the opportunity afforded by the presence of the constant term ($-\ln|a|$) in our solution to restate the constant of integration in order to avoid having to show this constant term explicitly in our solution at all.

Our solution can finally be rewritten by simply swapping around the terms inside the modulus brackets as follows:

$$\ln \left| x + \sqrt{x^2 + a^2} \right| + C'$$

and this matches the form of the solution I gave at the outset.

We can now see that the somewhat unusual standard integral we have obtained is a consequence first of the nature of the integral of $\sec x$, and second of the resolution of the compound trigonometric and inverse trigonometric functions which we obtained on reversing our substitution.

21 Three Standard Integrals Derived Using the Product-to-Sum Formulae

In the opening examples of [Part 3 Section 9](#) and the subsequent notes, I demonstrated how the trigonometric product-to-sum formulae could be used to allow us to integrate products of trigonometric functions. I will use these formulae here to derive three useful standard integrals which can be used to shortcut the process of applying the formulae if desired.

First, it is worth quoting the three formulae used in [Part 3](#) for easy reference. These are:

$$\sin \alpha \sin \beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$$

$$\cos \alpha \cos \beta = \frac{1}{2}(\cos(\alpha - \beta) + \cos(\alpha + \beta))$$

$$\sin \alpha \cos \beta = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta))$$

The three standard integrals we will evaluate contain in template form the pattern of the integrals evaluated in the first two examples of [Part 3 Section 9](#), and they are as follows:

$$I_1 = \int \sin mx \sin nx \, dx$$

$$I_2 = \int \cos mx \cos nx \, dx$$

$$I_3 = \int \sin mx \cos nx \, dx$$

We can use the three formulae we have quoted, each of which clearly corresponds closely with one of the above integrands, to restate the integrands as follows:

$$\sin mx \sin nx = \frac{1}{2}(\cos(m - n)x - \cos(m + n)x)$$

$$\cos mx \cos nx = \frac{1}{2}(\cos(m - n)x + \cos(m + n)x)$$

$$\sin mx \cos nx = \frac{1}{2}(\sin(m + n)x + \sin(m - n)x)$$

As a result of the above, we can restate each one of our three integrals in a form which can be easily evaluated, as follows:

$$I_1 = \frac{1}{2} \left(\int \cos(m - n)x \, dx - \int \cos(m + n)x \, dx \right)$$

$$I_2 = \frac{1}{2} \left(\int \cos(m - n)x \, dx + \int \cos(m + n)x \, dx \right)$$

$$I_3 = \frac{1}{2} \left(\int \sin(m + n)x \, dx + \int \sin(m - n)x \, dx \right)$$

Continuing in parallel, as we have begun, each of these integrates to the following preliminary solutions:

$$I_1 = \frac{1}{2} \left(\frac{1}{m - n} \sin(m - n)x - \frac{1}{m + n} \sin(m + n)x \right) + C$$

$$I_2 = \frac{1}{2} \left(\frac{1}{m - n} \sin(m - n)x + \frac{1}{m + n} \sin(m + n)x \right) + C$$

$$I_3 = \frac{1}{2} \left(-\frac{1}{m + n} \cos(m + n)x - \frac{1}{m - n} \cos(m - n)x \right) + C$$

Multiplying out the coefficients of 1/2 will give us final results which might allow easier calculation of the individual coefficients of the trigonometric functions. So we will do this and immediately write out the standard integrals we have thus derived:

$$\int \sin mx \sin nx \, dx = \frac{1}{2(m - n)} \sin(m - n)x - \frac{1}{2(m + n)} \sin(m + n)x + C$$

$$\int \cos mx \cos nx \, dx = \frac{1}{2(m - n)} \sin(m - n)x + \frac{1}{2(m + n)} \sin(m + n)x + C$$

$$\int \sin mx \cos nx \, dx = -\frac{1}{2(m+n)} \cos(m+n)x - \frac{1}{2(m-n)} \cos(m-n)x + C$$

22 The Strange Case of the Integral of $\sin x \cos x$

The very first integral we considered in this book, right at the beginning of [Part 1](#), was the following one:

$$I = \int \sin x \cos x \, dx$$

We obtained a solution to this integral using what was subsequently revealed to be a form of the method of substitution – the solution obtained was as follows:

$$I = \frac{1}{2} \sin^2 x + C$$

However, in [Part 3 Section 9](#) on trigonometric integrals, we used a trigonometric product-to-sum identity to obtain the following, apparently different, solution:

$$I = -\frac{1}{4} \cos 2x + C$$

Interestingly, we can produce yet a third solution to this integral. The first solution above, which we obtained at the beginning of [Part 1](#), was explained as being achieved by making the substitution $u = \sin x$. We can, however, also try the substitution $u = \cos x$, which has as its consequence:

$$\frac{du}{dx} = -\sin x \text{ so that } dx = -\frac{1}{\sin x} du$$

Making these substitutions into our integral, we have the following version in terms of u :

$$\int u \sin x \cdot \frac{-1}{\sin x} du = -\int u du$$

This integral in terms of u integrates to give the solution:

$$-\frac{1}{2} u^2 + C$$

and on reversing the substitution, we have our third solution in terms of x as follows:

$$I = -\frac{1}{2} \cos^2 x + C$$

We have now assembled a total of three seemingly quite different solutions to the same integral. Let's list them together, so that we can contemplate the true enormity of the situation. They are:

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$$\frac{1}{2}\sin^2x + C, -\frac{1}{2}\cos^2x + C \text{ and } -\frac{1}{4}\cos 2x + C$$

In fact, these three solutions can be shown to be different statements of the same solution. This can be done because the trigonometric identities establish some unexpected equivalences between apparently different expressions containing trigonometric functions.

Looking at the three solutions listed above, we might expect to call to our aid the Pythagorean identity:

$$\sin^2x + \cos^2x = 1$$

as well as the power-reduction formula:

$$\sin^2x = \frac{1}{2}(1 - \cos 2x)$$

The first of these can be rearranged to show that:

$$\cos^2x = 1 - \sin^2x \text{ so that } -\cos^2x = \sin^2x - 1$$

From this, multiplying both sides of the second equation by $1/2$, we have the immediate consequence for the second of our alternative solutions to the effect that:

$$-\frac{1}{2}\cos^2x = \frac{1}{2}\sin^2x - \frac{1}{2} \text{ or alternatively } \frac{1}{2}\sin^2x = -\frac{1}{2}\cos^2x + \frac{1}{2}$$

Furthermore, the power-reduction formula multiplies out and rearranges as follows:

$$\sin^2x = \frac{1}{2} - \frac{1}{2}\cos 2x \text{ so that } -\frac{1}{2}\cos 2x = \sin^2x - \frac{1}{2}$$

Multiplying both sides by $1/2$ once again, we have the following important consequence for our third solution:

$$-\frac{1}{4}\cos 2x = \frac{1}{2}\sin^2x - \frac{1}{4} \text{ or alternatively } \frac{1}{2}\sin^2x = -\frac{1}{4}\cos 2x + \frac{1}{4}$$

These manipulations should convince us that the three solutions we listed earlier only differ from one another by a constant quantity. From the reasoning based on the Pythagorean identity, we can state that:

$$\frac{1}{2}\sin^2x + C = -\frac{1}{2}\cos^2x + C' \text{ where } C' = C + \frac{1}{2}$$

and our use of the power-reduction formula gives us this:

$$\frac{1}{2}\sin^2 x + C = -\frac{1}{4}\cos 2x + C'' \text{ where } C'' = C + \frac{1}{4}$$

We can therefore say quite confidently that the three different solutions we have obtained for the integral in question are in fact equivalent solutions. This is possible in consequence of the action we have called *restating the constant* elsewhere in the text.

It is only when we evaluate an *indefinite* integral that a constant difference between two solutions seems to make a difference – and then, it is only in terms of how the solution is expressed. If we were to evaluate a *definite* integral of $\sin x \cos x$, we would find that the three solutions gave the same results.

To illustrate this, let us consider the following definite integral:

$$\int_0^{\pi/2} \sin x \cos x \, dx$$

The three “different” solutions we listed above give us the following results:

$$\left[\frac{1}{2} \sin^2 x \right]_0^{\pi/2} = \frac{1}{2} - 0 = \frac{1}{2}$$

$$\left[-\frac{1}{2} \cos^2 x \right]_0^{\pi/2} = 0 + \frac{1}{2} = \frac{1}{2}$$

$$\left[-\frac{1}{4} \cos 2x \right]_0^{\pi/2} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

23 Investigation of a Reduction Formula for the Integral of $\sin^m x \cos^n x$

In [Part 3 Section 9](#) there are several examples of the integration of trigonometric expressions of the form $\sin^m x \cos^n x$. It will be interesting to see whether we can derive some kind of reduction formula which relates the integrals of various expressions of this type for different values of m and n .

In the ensuing investigation, we will adopt the following notation for the general trigonometric integral of the type which interests us here:

$$T_{m,n} = \int \sin^m x \cos^n x \, dx$$

where I have chosen the letter T in recognition of the trigonometric nature of the integrand.

To start our investigation, it may be a good idea to look at cases with very small values for the placeholders m and n , since in a typical reduction formula, integrals with higher valued placeholders will be found to depend on others with lower valued placeholders.

Accordingly, we should perhaps start with the very lowest values for m and n , by looking at the following integral:

$$T_{1,1} = \int \sin x \cos x \, dx$$

Leaving aside the discussion of this integral in the preceding section, we can simply note that we evaluated this integral at the beginning of [Part 1](#), where the solution was found to be:

$$T_{1,1} = \frac{1}{2} \sin^2 x + C$$

Having made that fairly simple start, we can now take the next step of considering cases where one of the two identifying placeholders is kept to a value of 1, while the other is allowed to vary. First, we will consider the case of:

$$T_{m,1} = \int \sin^m x \cos x \, dx$$

We can deal with this by substitution – even though it is a general integral with a symbolic power of $\sin x$ present, we should be able to recognize the presence of a function-derivative pair.

I will carry out the substitution explicitly, instead of using the rule of integration based on substitution. We will substitute:

$$u = \sin x \text{ so that } \frac{du}{dx} = \cos x \text{ and } dx = \frac{1}{\cos x} du$$

Making the substitution therefore gets us this integral in terms of u :

$$\int u^m \cos x \frac{1}{\cos x} du = \int u^m du = \frac{1}{m+1} u^{m+1} + C$$

and reversing the substitution yields the following solution in terms of x :

$$T_{m,1} = \frac{1}{m+1} \sin^{m+1} x + C$$

We note that this is an explicit formula for the general integral we evaluated, and not a reduction formula in terms of a lower value for the placeholder m .

The logical; next step will be to consider the case where the first placeholder is fixed at a value of 1, and the second one is allowed to vary. Our integral this time will be:

$$T_{1,n} = \int \sin x \cos^n x \, dx$$

Once again, we can use substitution. This time, however, we will choose:

$$u = \cos x \text{ so that } \frac{du}{dx} = -\sin x \text{ and } dx = -\frac{1}{\sin x} du$$

Putting these facts back into the original integral gives us this integral in terms of u :

$$\int \sin x u^n \cdot \left(-\frac{1}{\sin x}\right) du = -\int u^n du = -\frac{1}{n+1} u^{n+1} + C$$

and reversing the substitution produces this solution in terms of x :

$$T_{1,n} = -\frac{1}{n+1} \cos^{n+1} x + C$$

It has been our experience in this book that the reduction formulae for integrals containing powers of trigonometric formulae frequently relate the integral of a particular power of the function to another integral containing a power which is lower than the one we started with by 2. In case we are confronted with a similar situation here, it will be useful to pay some attention to the integrals $T_{m,2}$ and $T_{2,n}$ before we proceed to the fully general case.

We will therefore start with:

$$T_{m,2} = \int \sin^m x \cos^2 x dx$$

Since we do not in this instance have a function-derivative pair to work with, we will attempt to integrate this case by parts. Before choosing parts, we should note from [Appendix 4 Section 2](#) that we have a standard integral for $\cos^2 x$ which is as follows:

$$\int \cos^2 x dx = \frac{1}{2}(x + \sin x \cos x) + C$$

I have chosen the version of this standard integral which avoids trigonometric functions of $2x$, so that we can limit ourselves to functions of x alone in our solution if at all possible. With this useful piece of information at our disposal, we can choose parts as follows:

$$u = \sin^m x \text{ and } \frac{dv}{dx} = \cos^2 x$$

These choices have the following consequences:

$$\frac{du}{dx} = m \sin^{m-1} x \cos x \text{ and } v = \frac{1}{2}(x + \sin x \cos x)$$

The integration by parts formula then gives us:

$$T_{m,2} = \frac{1}{2}(x \sin^m x + \sin^{m+1} x \cos x) - \frac{1}{2} m \int (x \sin^{m-1} x \cos x + \sin^m x \cos^2 x) dx$$

The integral on the right-hand side can be split into two components, as follows:

$$\int x \sin^{m-1} x \cos x dx + \int \sin^m x \cos^2 x dx = \int x \sin^{m-1} x \cos x dx + T_{m,2}$$

We can see that the integral we sought to evaluate has made its reappearance on the right-hand side, so that we might hope to be able to find its value by solving the resulting equation. We can therefore restate our earlier expression for $T_{m,2}$, obtained directly from the integration by parts formula, as follows – this time, multiplying out all the common coefficients to save us the trouble later:

$$T_{m,2} = \frac{1}{2}x\sin^m x + \frac{1}{2}\sin^{m+1}x \cos x - \frac{m}{2} \int x\sin^{m-1}x \cos x dx - \frac{m}{2}T_{m,2}$$

We will now abbreviate the integral on the right-hand side using the identifier I_1 , and move all the instances of $T_{m,2}$ to the left-hand side. We will then have:

$$\left(\frac{m}{2} + 1\right)T_{m,2} = \frac{1}{2}x\sin^m x + \frac{1}{2}\sin^{m+1}x \cos x - \frac{m}{2}I_1$$

where:

$$I_1 = \int x\sin^{m-1}x \cos x dx$$

We can now clear this equation up a little by simplifying the coefficient of $T_{m,2}$ on the left-hand side in the following way:

$$\frac{m}{2} + 1 = \frac{m}{2} + \frac{2}{2} = \frac{m+2}{2}$$

The intermediate integral I_1 is in prime condition for integration by parts, with the aid of the **LIATE** rule to help us make the best choices. We will choose parts as follows:

$$u = x \text{ and } \frac{dv}{dx} = \sin^{m-1}x \cos x$$

We might feel that we have made a dreadful error here, until we reflect that with the above choice of dv/dx , we can say the following:

$$\frac{du}{dx} = 1 \text{ and } v = \int \sin^{m-1}x \cos x dx = T_{m-1,1} = \frac{1}{m}\sin^m x$$

omitting the constant of integration for the time being, confident that it will be back in place at the end of the process.

The integration by parts formula now gives us the following expression for I_1 :

$$I_1 = \frac{1}{m}x\sin^m x - \frac{1}{m} \int \sin^m x dx$$

Now, we do not have a ready formula to hand for the integral of $\sin^m x$: however, we do have a reduction formula for this general integral, and we can use this formula repeatedly as many times as necessary to obtain an expression for the integral if we are given a particular value

for m . (The first six integrals in this sequence are listed based on repeated use of the reduction formula in [Appendix 4 Section 2](#).) We will therefore represent this integral using an abbreviation, as follows:

$$S_m = \int \sin^m x dx$$

so that we can now give the following expression for I_1 :

$$I_1 = \frac{1}{m} x \sin^m x - \frac{1}{m} S_m$$

Returning to our expression for $T_{m,2}$ with the aid of the above two pieces of work, we can now restate it as follows:

$$\left(\frac{m+2}{2}\right) T_{m,2} = \frac{1}{2} x \sin^m x + \frac{1}{2} \sin^{m+1} x \cos x - \frac{1}{2} x \sin^m x + \frac{1}{2} S_m$$

We can remove the coefficients of $1/2$ which are applied uniformly to both sides to obtain:

$$(m+2) T_{m,2} = x \sin^m x + \sin^{m+1} x \cos x - x \sin^m x + S_m$$

The two $x \sin^m x$ terms on the right-hand side are seen to cancel each other out, giving us the following:

$$(m+2) T_{m,2} = \sin^{m+1} x \cos x + S_m$$

and so we finally arrive at the following formula for $T_{m,2}$:

$$T_{m,2} = \frac{1}{m+2} \sin^{m+1} x \cos x + \frac{1}{m+2} S_m \text{ where } S_m = \int \sin^m x dx$$

We can now turn to the similar case of:

$$T_{2,n} = \int \sin^2 x \cos^n x dx$$

It should be possible, given the similarity in structure, to use the same methods we used to deal with $T_{m,2}$. This time, however, we will need a different standard integral to help us – to wit, the following one:

$$\int \sin^2 x dx = \frac{1}{2} (x - \sin x \cos x) + C$$

With this in mind, we will again use integration by parts, and we will choose parts as follows:

$$u = \cos^n x \text{ and } \frac{dv}{dx} = \sin^2 x$$

The consequences of these choices are:

$$\frac{du}{dx} = -n \sin x \cos^{n-1} x \text{ and } v = \frac{1}{2}(x - \sin x \cos x)$$

and the integration by parts formula now gives us:

$$T_{2,n} = \frac{1}{2}(x \cos^n x - \sin x \cos^{n+1} x) + \frac{1}{2}n \int (x \sin x \cos^{n-1} x - \sin^2 x \cos^n x) dx$$

(Note the plus sign just before the integral on the right-hand side, which is a consequence of the minus sign preceding the expression for du/dx .) Once again, the integral on the right-hand side splits up as follows:

$$\int x \sin x \cos^{n-1} x dx - \int \sin^2 x \cos^n x dx = \int x \sin x \cos^{n-1} x dx - T_{2,n}$$

Let's now put this back into our earlier expression for $T_{2,n}$ and, as we did in the previous case, multiply out all the coefficients. We obtain:

$$T_{2,n} = \frac{1}{2}x \cos^n x - \frac{1}{2}\sin x \cos^{n+1} x + \frac{n}{2} \int x \sin x \cos^{n-1} x dx - \frac{n}{2}T_{2,n}$$

We will now abbreviate the integral on the right-hand side as I_1 and move all instances of $T_{2,n}$ to the right-hand side, to restate the above equation in the following way:

$$\left(\frac{n}{2} + 1\right)T_{2,n} = \frac{1}{2}x \cos^n x - \frac{1}{2}\sin x \cos^{n+1} x + \frac{n}{2}I_1$$

where:

$$I_1 = \int x \sin x \cos^{n-1} x dx$$

As before, we now proceed to tidy up this equation. First, the coefficient of $T_{2,n}$ on the left-hand side simplifies as follows:

$$\frac{n}{2} + 1 = \frac{n}{2} + \frac{2}{2} = \frac{n+2}{2}$$

The intermediate integral I_1 can be dealt with by parts, with the following choices, based on the **LIATE** rule:

$$u = x \text{ and } \frac{dv}{dx} = \sin x \cos^{n-1} x$$

The consequences are:

$$\frac{du}{dx} = 1 \text{ and } v = \int \sin x \cos^{n-1} x dx = T_{1,n-1} = -\frac{1}{n} \cos^n x$$

and so the integration by parts formula gives us the following preliminary result:

$$I_1 = -\frac{1}{n}x\cos^n x + \frac{1}{n}\int \cos^n x dx$$

Treating this result in a similar way to the previous case, we will abbreviate the integral on the right-hand side above as follows:

$$C_n = \int \cos^n x dx$$

bearing in mind that we have a reduction formula from [Part 2 Section 13](#) of the main text, as well as a set of solutions for low values of n in [Appendix 4 Section 2](#). Our expression for I_1 now becomes:

$$I_1 = -\frac{1}{n}x\cos^n x + \frac{1}{n}C_n$$

We can now put the above information back into the expression for $T_{2,n}$ which we obtained above. It becomes:

$$\left(\frac{n+2}{2}\right)T_{2,n} = \frac{1}{2}x\cos^n x - \frac{1}{2}\sin x \cos^{n+1}x - \frac{1}{2}x\cos^n x + \frac{1}{2}C_n$$

Once again, we can remove the coefficients of $1/2$ which are applied throughout both sides to get the following equation:

$$(n+2)T_{2,n} = x\cos^n x - \sin x \cos^{n+1}x - x\cos^n x + C_n$$

The right-hand side terms in $x\cos^n x$ cancel each other out, and we can divide both sides by $n+2$ to arrive at the following expression for $T_{2,n}$:

$$T_{2,n} = -\frac{1}{n+2}\sin x \cos^{n+1}x + \frac{1}{n+2}C_n \text{ where } C_n = \int \cos^n x dx$$

We could, of course, express this instead as follows, getting rid of the opening minus sign:

$$T_{2,n} = \frac{1}{n+2}C_n - \frac{1}{n+2}\sin x \cos^{n+1}x$$

We have now obtained expressions (perhaps with varying degrees of usefulness) for the following instances of our general integral:

$$T_{1,1}, T_{m,1}, T_{m,2}, T_{1,n} \text{ and } T_{2,n}$$

This should be enough foundational work, and we can now proceed to look at what we can make of the general case, $T_{m,n}$

Let's start with a useful reminder to the effect that:

$$T_{m,n} = \int \sin^m x \cos^n x dx$$

In obtaining reduction formulae, we usually begin by an application of integration by parts. To facilitate a useful choice of parts, we will extract one factor of $\cos x$ from the integrand, and restate the integral as follows:

$$\int \sin^m x \cos^{n-1} x \cos x dx$$

This device allows us to choose parts in a useful way, as follows:

$$u = \sin^m x \cos^{n-1} x \text{ and } \frac{dv}{dx} = \cos x$$

The consequences are:

$$\frac{du}{dx} = m \sin^{m-1} x \cos^n x - (n-1) \sin^{m+1} x \cos^{n-2} x \text{ and } v = \sin x$$

The first of the above consequences was given by an application of the Product Rule to differentiate the expression we chose to be u , and the second shows the usefulness of the choice of parts we were enabled to make by restating the integrand.

We can now use the integration by parts formula to obtain the following expression for our general integral:

$$T_{m,n} = \sin^{m+1} x \cos^{n-1} x - \int (m \sin^m x \cos^n x + (n-1) \sin^{m+2} x \cos^{n-2} x) dx$$

We can, of course, divide the integral on the right-hand side into two separate integrals and take the coefficients in their integrands outside the integral sign. This gives us the following:

$$T_{m,n} = \sin^{m+1} x \cos^{n-1} x - m \int \sin^m x \cos^n x dx + (n-1) \int \sin^{m+2} x \cos^{n-2} x dx$$

Both the integrals on the right-hand side are now integrals of the type we are considering here, and as such they can be expressed using the subscripted- T notation we have adopted. Using this notation, our equation becomes

$$T_{m,n} = \sin^{m+1} x \cos^{n-1} x - m T_{m,n} + (n-1) T_{m+2,n-2}$$

Bringing all the terms in $T_{m,n}$ over to the left-hand side, we have:

$$(m+1) T_{m,n} = \sin^{m+1} x \cos^{n-1} x + (n-1) T_{m+2,n-2}$$

and if we now divide both sides by $m+1$, we will obtain the general formula we have been hoping for:

$$T_{m,n} = \frac{1}{m+1} \sin^{m+1} x \cos^{n-1} x + \frac{n-1}{m+1} T_{m+2,n-2}$$

In conclusion, it will be interesting to look at a list of the various formulae we have derived here. They are as follows:

$$T_{1,1} = \frac{1}{2} \sin^2 x + C$$

$$T_{m,1} = \frac{1}{m+1} \sin^{m+1} x + C$$

$$T_{1,n} = -\frac{1}{n+1} \cos^{n+1} x + C$$

$$T_{m,2} = \frac{1}{m+2} \sin^{m+1} x \cos x + \frac{1}{m+2} S_m \text{ where } S_m = \int \sin^m x dx$$

$$T_{2,n} = -\frac{1}{n+2} \sin x \cos^{n+1} x + \frac{1}{n+2} C_n \text{ where } C_n = \int \cos^n x dx$$

$$T_{m,n} = \frac{1}{m+1} \sin^{m+1} x \cos^{n-1} x + \frac{n-1}{m+1} T_{m+2,n-2}$$

For the integrals of powers of $\sin x$ and $\cos x$, we derived the following reduction formulae in [Part 2](#) of the main text:

$$S_n = \frac{n-1}{n} S_{n-2} - \frac{1}{n} \sin^{n-1} x \cos x$$

$$C_n = \frac{n-1}{n} C_{n-2} + \frac{1}{n} \sin x \cos^{n-1} x$$

We can see from this summary that the general integral $T_{m,n}$ has been found to depend on another general integral $T_{p,q}$ where p is *greater* than m , while q has the more usual relation found in a reduction formula of being less than n . This is a phenomenon we observed at the end of [Part 2](#), where we derived a reduction formula for the general integral known as the “beta function” $B(m,n)$. This last integral was, like our current trigonometric one, the integral of the product of two functions each raised to a power, so that we might be tempted to conclude that integrals of this type have a tendency to generate reduction formulae of the kind we have found in this case.

We should note further that the integrals $T_{m,2}$ and $T_{2,n}$ depend on the integrals of powers of $\sin x$ and $\cos x$. Since the general case $T_{m,n}$ depends on an instance of the T integral which differs in indexation by an even amount (actually, by 2 for each subscript!), we must conclude that any of these T integrals whose second subscript is even will ultimately itself depend on the integral of a power of $\sin x$. This is because the general reduction formula for $T_{m,n}$ is found to depend on $T_{m+2,n-2}$ - the second subscript on the right-hand side is 2 less than the second subscript on the left-hand side. Consequently, if we apply the reduction

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formula often enough for a version of our symbolic integral with an even value of n , we will eventually reach $T_{m,2}$, which we see above depends on the integral I have named S_n .

Index of Integrals

The following table lists every integral evaluated in the text and appendices, with a brief note on the method used in each case and a reference to the location of the full solution. Integrals containing explicit constants and coefficients are listed first, and symbolic integrals in a second table.

Sometimes, an integral will have been evaluated in two places in the text, perhaps using slightly different techniques; in these cases, two references to the text are given. Finally, I have also chosen to include some references to intermediate integrals evaluated as part of a larger integration problem if such intermediate integrals have been fully evaluated and they seem significant enough to act as useful reference material – in such cases, the following table shows the integrands as functions of the variable x rather than any substitution variable which may be used in the text.

The final section contains a list of the standard integrals which are discussed at various points in the text, and most of which are also listed with brief comments in [Appendix 4 Section 2](#).

1 Integrals with Explicit Constants

<i>Integral</i>	<i>Method of Evaluation</i>	<i>Location in Text</i>
$\int \frac{1}{x^3 - 1} dx$	Partial fractions	Part 3 Section 2 Example 3
$\int \frac{1}{1 + \sqrt{x}} dx$	Substitution	Part 1 Section 9 Example 1
$\int \frac{1}{\sqrt{x^2 + 1}} dx$	Substitution	Part 1 Section 7
$\int \frac{1}{x^{2/3} - x^{1/3}} dx$	Substitution	Part 1 Section 9 Example 3
$\int \frac{1}{\sqrt{4 - 2x - x^2}} dx$	Substitution	Part 1 Section 12 Example 1
$\int \frac{1}{x} \sqrt{\frac{1-x}{x}} dx$	Substitution	Part 3 Section 7 Example 1
$\int \frac{1}{\sqrt{5 + x^2}} dx$	Substitution	Part 1 Section 11 Example 2
$\int \frac{1}{\sqrt{9 - x^2}} dx$	Substitution	Part 1 Section 11 Example 1
$\int \frac{1}{\sqrt{x(1-x)}} dx$	Substitution	Part 1 Section 11 Example 5
$\int \frac{1}{\sqrt{x^2 - 4}} dx$	Substitution	Part 1 Section 11 Example 3
$\int \frac{1}{x^2 + 1} dx$	Substitution	Part 1 Section 12 Example 2 (intermediate integral I_1)
$\int \frac{1}{(x^2 - 4x + 5)^2} dx$	Substitution	Part 3 Section 2 Example 4 (intermediate integral I_5)
$\int \frac{1}{(x-1)(x^2 + x + 1)} du$	Partial fractions	Part 3 Section 8 Example 6 (intermediate integral I_2)

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$\int \frac{1}{(x-1)^2(x^2+x+1)^2} du$	Partial fractions	Part 3 Section 8 Example 6 (intermediate integral I_3)
$\int \frac{2+\sqrt{x}}{3-\sqrt{x}} dx$	Multiple techniques	Part 3 Section 6 Example 1
$\int \frac{3}{4+x^{1/3}} dx$	Multiple techniques	Part 3 Section 6 Example 2
$\int \frac{6}{(x-4)(x-1)} dx$	Partial fractions	Part 3 Section 1 Example 1
$\int \frac{12}{(x+1)^2} dx$	Rule of integration	Part 3 Section 2 Example 1(a) (intermediate integral I_3)
$\int \frac{29}{x^2-5} dx$	Partial fractions	Appendix 5 Section 18
$\int \frac{64}{x^4-16} dx$	Partial fractions	Part 3 Section 4 Example 3
$\int \frac{\sqrt{x}}{x-1} dx$	Multiple techniques	Part 3 Section 4 Example 1
$\int \frac{\sqrt{x}+1}{\sqrt{x}(\sqrt[3]{x}+1)} dx$	Multiple techniques	Part 3 Section 6 Example 3
$\int \frac{\sqrt{x+9}}{x} dx$	Substitution	Part 3 Section 7 Example 3
$\int \sqrt{4-2x-x^2} dx$	Substitution	Part 1 Section 12 Example 3
$\int \frac{\sqrt{4-x}}{x^2} dx$	Multiple techniques	Part 3 Section 4 Example 2
$\int \sqrt[3]{(16-x^2)} dx$	Substitution	Part 1 Section 11 Example 4
$\int \frac{\sqrt[3]{x}+\sqrt{x}}{\sqrt[3]{x}} dx$	Substitution Rearranged integrand	Part 1 Section 9 Example 4
$\int \sqrt[3]{\frac{x+1}{x-1}} dx$	Multiple techniques	Part 3 Section 8 Example 6
$\int \frac{x^{1/4}+5}{x-16} dx$	Multiple techniques	Part 3 Section 4 Example 3
$\int \frac{x}{x^2+1} dx$	Substitution	Part 1 Section 12 Example 2 (intermediate integral I_1)
$\int \frac{x}{x^2+2x+2} dx$	Substitution	Part 1 Section 12 Example 2
$\int \frac{x-1}{(x-2)(x^2-4x+5)^2} dx$	Partial fractions	Part 3 Section 2 Example 4
$\int \frac{x-2}{x^2-4x+5} dx$	Substitution	Part 3 Section 2 Example 4 (intermediate integral I_2)
$\int \frac{x-2}{(x^2-4x+5)^2} dx$	Rule of integration	Part 3 Section 2 Example 4 (intermediate integral I_4)
$\int \frac{x-3}{(x^2-4x+5)^2} dx$	Multiple techniques	Part 3 Section 2 Example 4 (intermediate integral I_3)

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$\int \frac{x+2}{x^2+x+1} dx$	Substitution	Part 3 Section 2 Example 3 (intermediate integral I_2)
$\int \frac{x+5}{(x-2)(x+1)^2} dx$	Partial fractions	Part 3 Section 2 Example 1
$\int \frac{3x+1}{x^2+2x+2} dx$	Substitution	Part 3 Section 5 Example 2 (intermediate integral I_1)
$\int \frac{x^2}{\sqrt{x^2-4}} dx$	Multiple techniques	Part 3 Section 8 Example 5
$\int \frac{x^3-1}{x+2} dx$	Polynomial division	Part 3 Section 5 Example 3
$\int \frac{x^3-5x^2+3x-15}{x^2+3} dx$	Polynomial division	Part 3 Section 5 Example 4
$\int \frac{x^4+4}{x^2-5} dx$	Polynomial division	Part 3 Section 5 Example 5
$\int \frac{x^7+5x^6+10x^5+\dots}{x^2+2x+2} dx$	Polynomial division	Part 3 Section 5 Example 2
$\int \frac{2x}{x^2+1} dx$	Rule of integration	Part 1 Section 7 Example 5
$\int \frac{2x+3}{x^2+4} dx$	Multiple techniques	Part 3 Section 2 Example 2
$\int \frac{2x+3}{x^2-9} dx$	Partial fractions	Part 3 Section 1 Example 3
$\int \frac{3x+1}{x^2+2x+2} dx$	Substitution	Part 3 Section 5 Example 2 (intermediate integral I_1)
$\int \frac{3x+2}{\sqrt{x-9}} dx$	Substitution	Part 1 Section 9 Example 2
$\int \frac{3x+5}{(2x+3)^3} dx$	Substitution	Part 1 Section 8 Example 4
$\int \frac{3x^3-5x^2+10x-3}{3x+1} dx$	Polynomial division	Part 3 Section 5 Example 1
$\int \frac{6x^2+x+13}{(x-1)(x^2+4)} dx$	Partial fractions	Part 3 Section 2 Example 2
$\int \frac{7x+5}{(x-1)(x+2)} dx$	Partial fractions	Part 3 Section 1 Example 2
$\int \frac{7x+19}{(x+1)^2} dx$	Substitution	Part 3 Section 2 Example 1(b) (intermediate integral I_2)
$\int (2x+5)^3 dx$	Rule of integration	Part 1 Section 7
$\int (2x-1) \ln(x^2+1) dx$	Integration by parts	Part 2 Section 7 Example 1
$\int x\sqrt{1-3x^2} dx$	Substitution	Part 1 Section 3 Example 4
$\int x(3x+5)^2 dx$	Integration by parts	Part 2 Section 4 Example 3
$\int x(3x+5)^2 dx$	Substitution	Appendix 5 Section 7
$\int (x-2)^4 dx$	Rule of integration	Part 1 Section 7

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$\int x(x+1)^3 dx$	Substitution	Part 1 Section 8 Example 1
$\int x(x^2+1)^2 dx$	Substitution	Part 1 Section 3 Example 3
$\int xe^x dx$	Integration by parts	Part 2 Section 4 Example 1
$\int \frac{3}{2}xe^{2x} dx$	Integration by parts	Part 2 Section 5 Example 2 (intermediate integral I_2)
$\int x \cos x dx$	Integration by parts	Part 2 Section 3 Part 2 Section 6 Example 3
$\int x \sin^2 x dx$	Integration by parts	Part 3 Section 9 Example 12
$\int x \sin^2 x \cos^3 x dx$	Rule of integration	Part 3 Section 9
$\int x \tan^{-1} x dx$	Integration by parts	Part 2 Section 4 Example 4
$\int x \ln x dx$	Integration by parts	Part 2 Section 4 Example 5
$\int (x+1)\sqrt{(x+2)} dx$	Substitution	Part 1 Section 8 Example 3 Appendix 5 Section 1
$\int (x+2)(x+3)^4 dx$	Substitution	Part 1 Section 8 Example 2
$\int (3x+5) \cos \frac{1}{4}x dx$	Integration by parts	Part 2 Section 4 Example 2
$\int (3x^2-4) \tan^{-1} x dx$	Integration by parts	Part 2 Section 7 Example 2
$\int \sqrt{5+\sqrt{x}} dx$	Substitution	Part 1 Section 10 Example 1
$\int \sqrt{2+\sqrt{4+\sqrt{x}}} dx$	Substitution	Part 1 Section 10 Example 3
$\int \sqrt[3]{1+\sqrt{x-3}} dx$	Substitution	Part 1 Section 10 Example 2
$\int x^2(x^3+1)^4 dx$	Substitution	Part 1 Section 6 (treated as a definite integral)
$\int \frac{x^2}{1+x^2} dx$	Substitution	Appendix 5 Section 8
$\int \frac{x^2}{x^3-3} dx$	Rule of integration	Part 1 Section 7 Example 6
$\int \frac{x^2+x-1}{x(x^2-1)} dx$	Partial fractions	Part 3 Section 1 Example 4
$\int 3x^2(x^3-5)^4 dx$	Rule of integration	Part 1 Section 7 Example 2
$\int \frac{3}{2}x^2e^{2x} dx$	Integration by parts	Part 2 Section 5 Example 2 (intermediate integral I_1)
$\int x^2 \sin x dx$	Integration by parts	Part 2 Section 5 Example 1 Part 2 Section 6 Example 1
$\int \frac{x^3}{x^2+1} dx$	Substitution	Part 1 Section 8 Example 4 Part 1 Section 8 Example 5

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$\int \frac{x^3}{(x^3 - 1)^2} dx$	Partial fractions	Part 3 Section 8 Example 6 (intermediate integral I_1)
$\int x^3(x^4 + 2)^2 dx$	Rule of integration	Part 1 Section 7 Example 3
$\int \frac{20x^3}{x^4 - 16} dx$	Rule of integration	Part 3 Section 4 Example 3 (intermediate integral I_2)
$\int x^3 e^{2x} dx$	Integration by parts	Part 2 Section 5 Example 2 Part 2 Section 6 Example 2
$\int x^3 e^{2x} dx$	Reduction formula	Part 2 Section 13 Example 2
$\int x^3 \cos x dx$	Integration by parts	Part 2 Section 5 Example 3 Part 2 Section 6
$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$	Rule of integration	Part 3 Section 7 Example 5
$\int \frac{e^x}{(e^x - 1)(e^x + 3)} dx$	Multiple techniques	Part 3 Section 8 Example 1
$\int e^x \sin x dx$	Integration by parts	Part 2 Section 12 Example 1
$\int e^{2x} \cos 3x dx$	Integration by parts	Part 2 Section 12 Example 2
$\int \sqrt{e^x + 1} dx$	Substitution	Part 3 Section 7 Example 2
$\int \frac{1}{e^x + 1} dx$	Multiple techniques	Part 3 Section 8 Example 2
$\int \cos(x + 3) dx$	Rule of integration	Part 1 Section 7 Example 7
$\int \cos x (1 + \sin x + \sin^2 x) dx$	Substitution	Part 1 Section 3 Example 1
$\int \cos 2x dx$	Rule of integration	Part 1 Section 7 Example 8
$\int \cos(2x + 3) dx$	Substitution	Part 1 Section 3 Example 5
$\int \frac{\cos x}{\sin x} dx$	Rule of integration	Part 1 Section 7 Example 4
$\int \cos^7 x dx$	Reduction formula	Part 2 Section 13
$\int \cot x dx$	Rule of integration	Part 1 Section 7 Example 4
$\int \sec x dx$	Substitution	Appendix 5 Section 6
$\int \sec x \tan^2 x dx$	Integration by parts	Appendix 5 Section 12
$\int \frac{\sec^2 x}{\tan^3 x - \tan^2 x} dx$	Multiple techniques	Part 3 Section 8 Example 3
$\int \sec^3 x dx$	Integration by parts	Part 2 Section 12 Example 3
$\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$	Substitution	Part 1 Section 3 Example 6 Part 1 Section 9
$\int \sin \sqrt{x - 1} dx$	Multiple techniques	Part 3 Section 8 Example 4

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$\int (\sin x + \cos x)(\sin x - \cos x)^2 dx$	Substitution	Part 1 Section 3 Example 2
$\int \sin x \cos x dx$	Substitution	Part 1 Section 1 & Section 2
$\int \sin x \cos x dx$	Substitution	Appendix 5 Section 22
$\int \sin x \cos x dx$	Trigonometric identity	Appendix 5 Section 22
$\int \sin 3x \cos 5x dx$	Trigonometric identity	Part 3 Section 9 Example 1
$\int \sin 7x \sin 5x dx$	Trigonometric identity	Part 3 Section 9 Example 2
$\int \sin^2 x \cos^3 x dx$	Substitution	Part 3 Section 9 Example 3
$\int \sin^2 x \cos^4 x dx$	Multiple techniques	Part 3 Section 9 Example 6
$\int \sin^2 x \cos^5 x dx$	Substitution	Part 3 Section 9 Example 5
$\int \sin^3 x \cos x dx$	Rule of integration	Part 1 Section 7 Example 1
$\int \sin^3 x \cos^8 x dx$	Substitution	Part 3 Section 9 Example 4
$\int \sin^4 x \cos^6 x dx$	Multiple techniques	Part 3 Section 9 Example 7
$\int \sqrt{\frac{\sin^{-1} x}{1 - x^2}} dx$	Rule of integration	Part 3 Section 7 Example 4
$\int \tan^2 x \sec^3 x dx$	Trigonometric identity	Part 3 Section 9 Example 11
$\int \tan^3 x \sec^2 x dx$	Substitution	Part 3 Section 9 Example 8
$\int \tan^3 x \sec^4 x dx$	Substitution	Part 3 Section 9 Example 9
$\int \tan^5 x \sec^3 x dx$	Substitution	Part 3 Section 9 Example 10
$\int \tan^{-1} x dx$	Integration by parts	Part 2 Section 8 Example 2
$\int \ln x dx$	Integration by parts	Part 2 Section 8 Example 1
$\int \sqrt{\frac{\ln x + \sqrt{x^2 + 1} }{x^2 + 1}} dx$	Substitution	Part 3 Section 7 Example 7
$\int \frac{\sqrt[3]{1 + \ln x}}{x} dx$	Rule of integration	Part 3 Section 7 Example 6

2 Symbolic Integrals

Integral	Method of Evaluation	Location in Text
$\int \frac{1}{x+k} dx$	Rule of integration	Part 1 Section 7
$\int \frac{1}{(x+a)(x+b)} dx$	Partial fractions	Appendix 5 Section 17
$\int \frac{1}{ax+b} dx$	Rule of integration	Part 1 Section 7
$\int \frac{1}{(ax+b)(cx+d)} dx$	Partial fractions	Appendix 5 Section 17
$\int \frac{1}{\sqrt{(x^2+a^2)}} dx$	Substitution	Appendix 5 Section 20
$\int \frac{x}{\sqrt[3]{ax+b}} dx$	Substitution	Appendix 5 Section 5
$\int \frac{1}{x^2-a^2} dx$	Partial fractions	Appendix 5 Section 17
$\int \frac{x^n}{x^2+1} dx$	Reduction formula	Appendix 5 Section 9
$\int \frac{x^{2n-1}}{x^n+a} dx$	Substitution	Part 1 Section 8
$\int \frac{ax+b}{(cx+d)(ex+f)} dx$	Partial fractions	Appendix 5 Section 17
$\int \frac{px+q}{x^2+bx+c} dx$	Multiple techniques	Appendix 5 Section 19
$\int \frac{ax^{2n-1}}{bx^n+c} dx$	Substitution	Appendix 5 Section 4
$\int (x+k)^n dx$	Rule of integration	Part 1 Section 7
$\int (x+a)(x+b)^p dx$	Substitution	Appendix 5 Section 2
$\int (ax+b)^n dx$	Rule of integration	Part 1 Section 7
$\int (ax+b)(cx+d)^p dx$	Substitution	Appendix 5 Section 3
$\int x^{m-1}(1-x)^{n-1} dx$	Reduction formula	Part 2 Section 13 Example 4 (treated as a definite integral)
$\int x^n e^{2x} dx$	Reduction formula	Part 2 Section 13 Example 2
$\int e^{ax} \cos bx dx$	Integration by parts	Appendix 5 Section 10
$\int \cos mx \cos nx dx$	Trigonometric identity	Appendix 5 Section 21
$\int \cos^n x dx$	Reduction formula	Part 2 Section 13
$\int \sec^n x dx$	Reduction formula	Appendix 5 Section 11
$\int \sin mx \cos nx dx$	Trigonometric identity	Appendix 5 Section 21
$\int \sin mx \sin nx dx$	Trigonometric identity	Appendix 5 Section 21

$\int \sin^n x dx$	Reduction formula	Part 2 Section 13 Example 1
$\int \sin^m x \cos^n x dx$	Reduction formula	Appendix 5 Section 23
$\int \ln^n x dx$	Reduction formula	Part 2 Section 13 Example 3
$\int f(x+k) dx$	Substitution	Part 1 Section 7
$\int f(kx) dx$	Substitution	Part 1 Section 7
$\int (f(x))^n f'(x) dx$	Substitution	Part 1 Section 7
$\int \frac{f'(x)}{f(x)} dx$	Substitution`	Part 1 Section 7

3 Summary of Standard Integrals

<i>Algebraic Integrals</i>
$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + C$ $\int -\frac{1}{\sqrt{a^2 - x^2}} dx = \cos^{-1} \frac{x}{a} + C$ $\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left x + \sqrt{x^2 + a^2} \right + C$ $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$ $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln \left \frac{x-a}{x+a} \right + C$ $\int \sqrt{a^2 - x^2} dx = \frac{1}{2} \left(a^2 \sin^{-1} \frac{x}{a} + x \sqrt{a^2 - x^2} \right) + C$ $\int \sqrt{x^2 - a^2} dx = \frac{1}{2} \left(x \sqrt{x^2 - a^2} - a^2 \ln \left x + \sqrt{x^2 - a^2} \right \right) + C$ $\int \sqrt{x^2 + a^2} dx = \frac{1}{2} \left(x \sqrt{x^2 + a^2} + a^2 \ln \left x + \sqrt{x^2 + a^2} \right \right) + C$ $\int \frac{x}{x^2 + 1} dx = \frac{1}{2} \ln(x^2 + 1) + C$ $\int \frac{x^2}{x^2 + 1} dx = x - \tan^{-1} x + C$

$$\int \frac{x^3}{x^2 + 1} dx = \frac{1}{2}(x^2 - \ln(x^2 + 1)) + C$$

$$\int \frac{x^2}{x^2 + a^2} dx = x - \frac{1}{a} \tan^{-1} \frac{1}{a} x + C$$

$$\int \frac{x + a}{x + b} dx = x - (b - a) \ln|x + b| + C$$

$$\int \frac{1}{(x + a)(x + b)} dx = \frac{1}{a - b} \ln \left| \frac{x + b}{x + a} \right| + C$$

$$\int \frac{px + q}{x^2 + bx + c} dx = \frac{p}{2} \ln|x^2 + bx + c| - \frac{bp - 2q}{\sqrt{4c - b^2}} \tan^{-1} \frac{2}{\sqrt{4c - b^2}} \left(x + \frac{b}{2} \right) + C$$

Trigonometric Integrals

$$\int \sin kx dx = -\frac{1}{k} \cos kx + C$$

$$\int \cos kx dx = \frac{1}{k} \sin kx + C$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C$$

$$\int \cot x dx = \ln|\sin x| + C$$

$$\int \sin^2 x dx = \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) + C = \frac{1}{2} (x - \sin x \cos x) + C$$

$$\int \cos^2 x dx = \frac{1}{2} \left(x + \frac{1}{2} \sin 2x \right) + C = \frac{1}{2} (x + \sin x \cos x) + C$$

$$\int \tan^2 x dx = \tan x - x + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \tan^{-1} x dx = x \tan^{-1} x - \frac{1}{2} \ln(x^2 + 1) + C$$

$$\int \sin mx \sin nx dx = \frac{1}{2(m - n)} \sin(m - n)x - \frac{1}{2(m + n)} \sin(m + n)x + C$$

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$$\int \cos mx \cos nx \, dx = \frac{1}{2(m-n)} \sin(m-n)x + \frac{1}{2(m+n)} \sin(m+n)x + C$$

$$\int \sin mx \cos nx \, dx = -\frac{1}{2(m+n)} \cos(m+n)x - \frac{1}{2(m-n)} \cos(m-n)x + C$$

Exponential & Logarithmic Integrals

$$\int e^{kx} \, dx = \frac{1}{k} e^{kx} + C$$

$$\int \ln x \, dx = x \ln x - x + C$$